Analytic Combinatorics in Several Variables (2nd edition!)

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Smith Discrete Math Workshop
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What is analytic combinatorics?

- Encode an enumerative sequence of interest (often defined recursively) via its generating function, which is often rational or algebraic.
- Express the terms using the Cauchy Integral Formula:

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(z)\, d\mathbf{z}.$$

- The location of dominant points of the singular variety $\mathcal{V}$ of the GF determines exponential growth rate of coefficients.
- The type of singularity determines the asymptotic expansion, and can be analysed using stationary phase methods for oscillatory integrals.

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The univariate case is quite well understood

- Consider the Fibonacci numbers defined by the usual recurrence relation $a_n = a_{n-1} + a_{n-2}, a_0 = 0, a_1 = 1$.
- Using the generating function $F(x) = \sum_{n \geq 0} a_n x^n$, we translate into a defining equation for the GF, namely

$$F(x) = x + x(F(x) - a_0) + x^2 F(x)$$

and a solution

$$F(x) = \frac{x}{1 - x - x^2}.$$

- Partial fractions and a basic lookup table now give

$$a_n = \frac{1}{\sqrt{5}} \left(\theta^n - (-\theta)^{-n} \right) \sim \frac{1}{\sqrt{5}} \theta^n.$$

- Alternatively we could use residue theory near the dominant pole at $z = 1/\theta$. This generalizes better to higher dimensions.
- Book of Flajolet & Sedgewick (2009) summarizes univariate theory very well.

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Multivariate case

- Deriving multivariate GFs is often not much harder than in the univariate case.
- However analysing them *is* much harder, even for rational functions:
  - More ways to go to infinity!
  - Algebra: partial fraction decomposition does not apply in general.
  - Geometry: the singular variety $\mathcal{V}$ does not consist of isolated points, and may self-intersect.
  - Topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated.
  - Analysis: the (Leray) residue formula is much harder to use, and we still need to do integrals.
  - Computation: harder owing to the curse of dimensionality.

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Example (Delannoy walks)

- We count walks on the lattice $\mathbb{Z}^2$ starting at $(0,0)$ and ending at (r,s), with each step chosen from $\{\uparrow, \rightarrow, \nearrow\}$.
- The recurrence is

$$a_{rs} = \begin{cases} a_{r-1,s-1} + a_{r-1,s} + a_{r,s-1} & \text{if } r > 1, s > 1 \\ 1 & (r,s) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

  and GF is $(1 - x - y - xy)^{-1}$.
- How to compute asymptotics for $a_{r,s}$ for large r, s?

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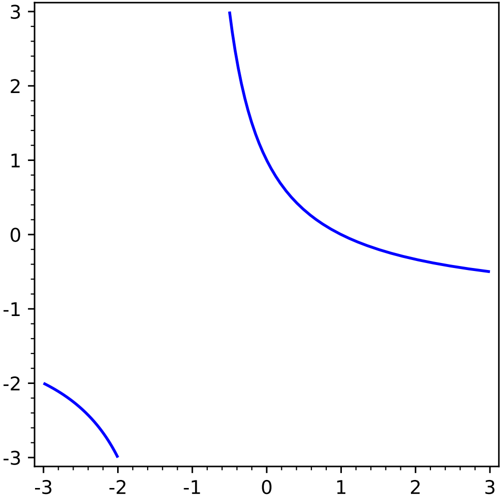
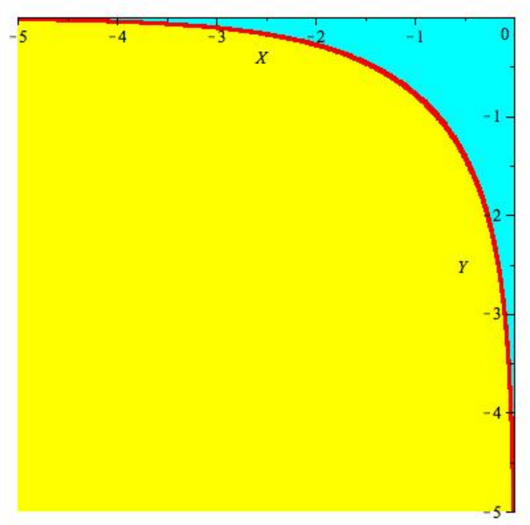
Example (Delannoy walks: singular variety)

The complex curve given by $1 - x - y - xy = 0$ (real points shown).



Example (Delannoy walks: singular variety in log coordinates)



Example (lemniscate)

- Consider $F = 1/H$ where

  $$H(x,y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

  This has positive coefficients, and H is a globally irreducible polynomial.
- All points except $(1,1)$ are smooth, and $(1,1)$ is a transverse strictly minimal double point.
- Note that H factors locally at $(1,1)$ but not globally. Analysis is essential - algebra is not enough.

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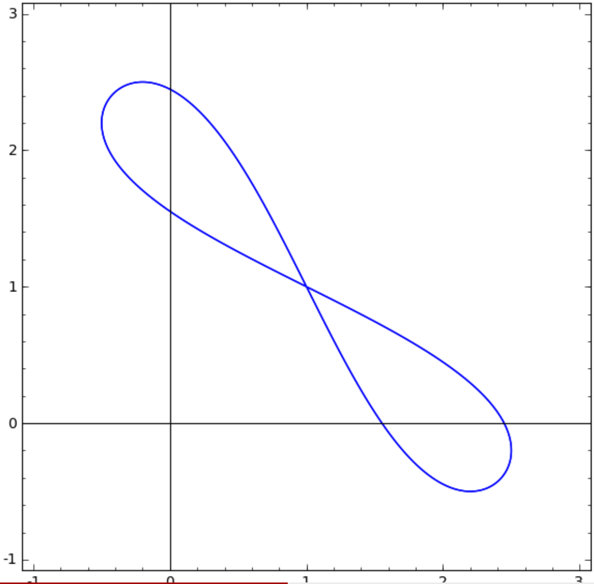

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- Robin Pemantle and I started working on this late last century, and various students, postdocs and other colleagues became involved. See `https://acsvproject.org`
- The first edition of our book ("ACSV") came out in 2013, after several papers, and was a natural stopping point.
- Steve Melczer (Waterloo) has rejuvenated the whole project.
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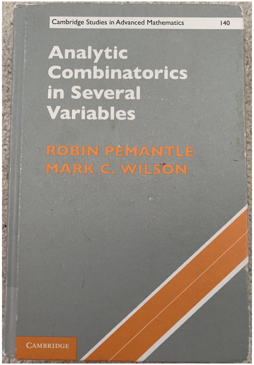

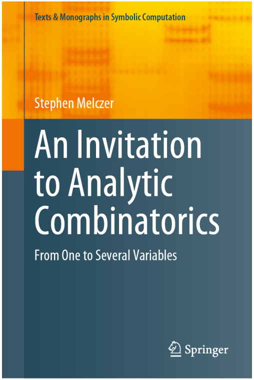

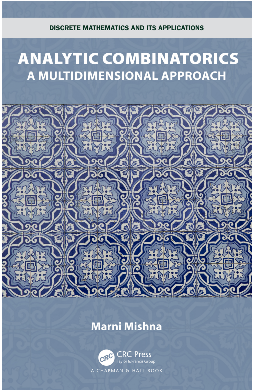

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- More rigorous
  - Rigorous results via stratified Morse theory (previously just intuition).
  - Better material on computational algebra; key formulae checked via Sage (previously there were many errors).
- More general
  - Many results extended to Laurent series (previously just power series)
  - Weaker hypotheses in some key theorems
- More usable
  - Self-contained theorems that can be applied without full understanding (previously just methods of solution).
- More accessible
  - More intuition and better pictures, some new examples, many more exercises than first edition.
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Some combinatorial examples analysed in the book

- Horizontally convex polyominoes
- Quantum random walk
- Lattice paths constrained in a quadrant
- Littlewood-Richardson coefficients
- Forests of trees with restricted offspring sizes
- Sequences with maximum number of distinct subsequences
- Aztec and fortress tilings
- Vector partitions
- Riordan arrays and Lagrange inversion

Delannoy walk asymptotics

- Uniformly for $r/s, s/r$ away from 0

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r+s-\Delta)^2}}.$$

where $\Delta = \sqrt{r^2 + s^2}$.

- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.
- Vastly many problems involving walks, sequences, sums of IID random variables are of similar difficulty level.
- If you know of the "diagonal method" (e.g. in Stanley EC1), note that it fails in almost all such cases.

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Delannoy walk numerics

Extracting a given diagonal is now easy: for example $a_{13n,19n} \sim AC^n n^{-1/2}$ where

$$C = \frac{5992140640923259546229532908662179346 87}{(\sqrt{530} - 13)^{19}(\sqrt{530} - 19)^{13}}$$
$$\approx 7.98556079229048 \times 10^{11}$$
$$A \approx 0.145544855274974.$$

- The relative error even for $n = 1$ is under 1% and error decreases as $1/n$.
- The second order approximation includes an $n^{-3/2}$ term and has relative error of order n^{-2}, etc.

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Example (Narayana numbers)

- These count rooted ordered trees by edges and leaves.
- The bivariate GF $w := F(x, y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(w, x, y) := w^2 - w\,[1 + x(y-1)] + xy = 0.$

- Using a known construction (Furstenberg) we obtain the rational GF

$$G(u, x, y) = \frac{u(1 - 2u - ux(1-y))}{1 - u - xy - ux(1-y)}.$$

such that $b_{rrs} = a_{rs}$.

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Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way, but the first term in the series is zero, so we need to go further. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

and we obtain asymptotics starting with s^{-2}. For example

$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$

- This example shows:
  - we can go beyond rational and even meromorphic GFs;
  - we may need to consider noncombinatorial GFs and higher order terms;
  - a genuinely multivariate theory is useful.

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Example (Infinite products: Quivers and Littlewood-Richardson coefficients)

- Consider the meromorphic non-rational generating function

$$F(x,y) = \prod_{i=1}^{\infty} \left(1 - x^i - y^i\right)^{-1}$$

arising in the study of chiral operators in 4-dimensional quiver gauge theories and in the enumeration of *Littlewood-Richardson coefficients* $c_{\mu\nu}^{\lambda}$.

- Write $F(x,y) = P(x,y)/(1-x-y)$ where $P(x,y) = \prod_{i=2}^{\infty} \left(1 - x^i - y^i\right)^{-1}$. It is actually easy to compute

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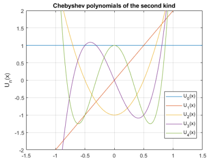
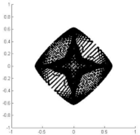
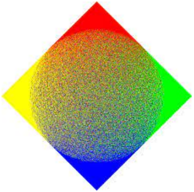
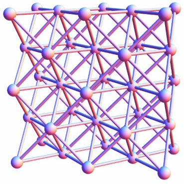
Some harder applications

- Chebyshev polynomials (smooth, two contributing critical points, periodicity)
- quantum random walks (smooth, non-isolated critical points)
- cube groves, frozen regions for tiling models in statistical mechanics (cone singularities)
- lattice Green's functions (non-transverse multiple points - don't know how to do yet!)



Publication reform

- Pressure is building for complete conversion of the journal system to open access (e.g. Plan S from European research funders)
- Large commercial publishers have incentives not aligned with scholarship or the interests of readers and authors, and provide overall low quality service for very high prices.
- The journal market is dysfunctional (not properly competitive).
- I am associated with several organizations aiming to improve this: MathOA, Free Journal Network, Publishing Reform Forum. If you would like to help or learn more, please contact me.
- Support the journals Algebraic Combinatorics (NOT the zombie J. Alg Comb) and Combinatorial Theory (NOT the zombie JCTA).