Arrow's Theorem

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HCSSIM Prime Time Theorem
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Motivating example

- Suppose that after you have heard all the Prime Time talks this summer, you each rank them from most to least interesting.
- Suppose you, as a group, then want to rank them in a single list, to summarize the group's opinion.
- Q: Which is the best way to do this? What does that even mean? What properties should a group ranking have?
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Preferences

- There is a set V of n voters and a set A of m alternatives that they can rank.
- Each voter has a complete strict ranking of all alternatives, from top to bottom choice.
- We write $i > j$ to mean that for the given voter, i is strictly preferred to j.
- Putting all these together gives us a preference profile.
- Formally, a profile is a function from V to $L(A)$, where $L(A)$ is the set of all possible rankings of A.
- Q: how many possible preference orders are there? how many profiles? how big is this for $m = 3, n = 2$?

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Profile example

Example

The alternatives are pizza toppings: anchovy, ham, pineapple. The voters are Dweezil, Ahmet, Diva, and Moon. One day their profile is:

Dw	A	D	M
ham	pineapple	anchovy	pineapple
anchovy	anchovy	ham	ham
pineapple	ham	pineapple	anchovy

A year later it is:

Dw	A	D	M
ham	anchovy	anchovy	pineapple
pineapple	pineapple	pineapple	ham
anchovy	ham	ham	anchovy

Social welfare function

- A social welfare function (SWF) is a function that takes each profile and outputs a societal ranking, which is just an element of $L(A)$.
- We write $i \succ j$ to mean that i is strictly preferred to j in the societal ranking.
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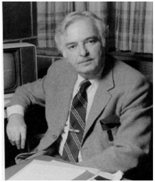
Examples of social welfare functions

There are many, some quite weird. For example:

- (dictatorship) Choose one voter and output their ranking.
- (Kemeny) If everyone has the same (unanimous) ranking, return that one. Otherwise return the unanimous profile that is closest (in some well defined sense) to the input.
- (Borda) Give m points each time an alternative is ranked first, $m-1$ points each time it is ranked second, ..., 1 point if ranked last. Rank in decreasing order of total score.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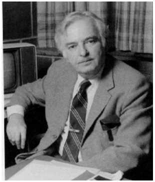


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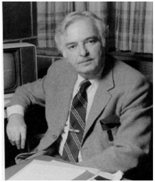

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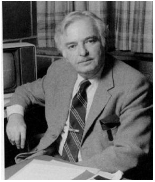


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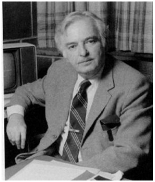


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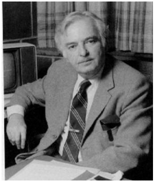


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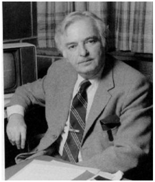


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Useful terminology

- For a fixed pair j, k of distinct alternatives, a profile is (j, k)-unanimous if all voters agree $j > k$.
- For a fixed pair j, k of distinct alternatives, two profiles are $\{j, k\}$-equivalent if for each voter, the relative ranking of j and k is the same in each profile.
- Q: Check these on the pizza topping example above. Which are satisfied?
- For a fixed pair j, k of distinct alternatives, a voter v is $\{j, k\}$-decisive if the relative ranking of j and k in the societal ranking always agrees with v's ranking.
- A dictator is a voter who is decisive over *all* pairs of alternatives.

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Desirable properties of a SWF

- Unanimity: for each j, k, if the profile is (j, k)-unanimous then the SWF ranks $j \succ k$.
- In other words, if all voters agree that $j > k$, then we must have $j \succ k$.
- Independence of Irrelevant Alternatives (IIA): for each j, k, if two profiles are $\{j, k\}$-equivalent, then in the societal ranking the relative ranking of j and k is the same.
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IIA example

- After finishing dinner, Professor X decides to order dessert. The waiter tells her there are two choices: apple pie and blueberry pie.
- Professor X orders the apple pie. After a few minutes the waiter returns and says that they also have cherry pie.
- Professor X says "In that case I'll have the blueberry pie."
- This seems unreasonable!

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The simple case $m = 2$

- In this case we are really just voting for the top societal alternative.
- We can use the usual majority rule: whichever alternative is top-ranked more often.
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Theorem (Arrow, 1951)

Suppose that $m \geq 3$ and $n \geq 2$. Then every social welfare function that satisfies both Unanimity and IIA is a dictatorship.

This was a big surprise when first proved! The proof I present was published by Yu (2012), simplifying previous proofs.

Proof stage 1: Pivotal voter exists, for each pair of alternatives

- Order the voters in some fixed way $v_1, \ldots, v_n$ and consider an arbitrary pair of distinct alternatives i, j.
- Choose any (i, j)-unanimous profile. By Unanimity, $i \succ j$ in the societal ordering.
- Swap i and j in each voter's order in turn. After all have been done, the societal order says $j \succ i$, by Unanimity.
- The first voter for which the societal ordering of i and j flips is called pivotal for (i, j).
- By IIA, it doesn't matter which (i, j)-unanimous profile we use — the same voter, say v, is found each time. Call this voter's position n_{ij}.

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Proof stage 2: four profiles

- Choose a third alternative k, and start with any profile P. Without loss of generality, assume that $j > k$ in v's ranking.
- Profile P is $\{j, k\}$-equivalent to a profile P' ranking i at the bottom for all voters strictly before v, i at the top for all voters after v, and i in the middle for v.
- Profile P' is $\{i, k\}$-equivalent to a profile P'' ranking j at the top for all voters strictly before v, and ranking j in the middle for voters v and later.
- Profile P''' is obtained from P'' by swapping j and k for voters strictly before v.

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The four profiles

$<$	v	$>$		$<$	v	$>$		$<$	v	$>$		$<$	v	$>$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	i		j	$\vdots$	i		k	$\vdots$	i
	j				j				i			i	i	
$\vdots$	$\vdots$	$\vdots$	$\rightarrow$	$\vdots$	$\vdots$	$\vdots$	$\rightarrow$	$\vdots$	$\vdots$	$\vdots$	$\rightarrow$	$\vdots$	$\vdots$	$\vdots$
	k				i				j	j			j	j
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
				i	k			i	k	k		j	k	k

Proof stage 3: pivotal voter for (i, j) is decisive over $\{j, k\}$

- In P'', we must have $j \succ k$ by unanimity. We also have $i \succ j$ since v is pivotal for (i, j). Thus $i \succ k$.
- Thus in P' we have $j \succ i$ because v is pivotal, and $i \succ k$ by $\{i, k\}$-equivalence. Thus $j \succ k$.
- Hence in P, $j \succ k$.
- Since P was arbitrary, v is decisive over $\{j, k\}$.
- Hence $j \succ k$ in P''' so we have not yet reached the pivotal voter for (j, k) — in other words, $n_{jk} \geq n_{ij}$.

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Proof stage 3: pivotal voter for (i, j) is decisive over $\{j, k\}$

- In P'', we must have $j \succ k$ by unanimity. We also have $i \succ j$ since v is pivotal for (i, j). Thus $i \succ k$.
- Thus in P' we have $j \succ i$ because v is pivotal, and $i \succ k$ by $\{i, k\}$-equivalence. Thus $j \succ k$.
- Hence in P, $j \succ k$.
- Since P was arbitrary, v is decisive over $\{j, k\}$.
- Hence $j \succ k$ in P''' so we have not yet reached the pivotal voter for (j, k) — in other words, $n_{jk} \geq n_{ij}$.

Proof stage 4: Pivotal voter is a dictator

- Since i, j, k are arbitrary, we have for all distinct i, j, k

$$n_{jk} \geq n_{ij} \geq n_{ki} \geq n_{jk}.$$

- Thus all the n_{ij} are equal, and hence v is pivotal for all pairs of alternatives.
- Thus v is decisive over all pairs of alternatives, and is hence a dictator.
- This ends the proof!

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Where to from here?

- Arrow was the youngest Nobel Economics winner, largely for this result.
- Arrow's Theorem leads quickly to the Gibbard-Satterthwaite theorem, which says that if we are choosing a unique winner instead of ranking, and every candidate can win in some situation, then the only way to avoid incentives for strategic voting is to have a dictator.
- Many people were shocked by such results, believing that they make democracy impossible.
- Eventually it was realized that maybe IIA is not such a reasonable assumption.
- We can try to use more information (how much does each voter like a given alternative) rather than just a ranking, but that causes other problems.

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Further reading

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