Power measures and the sequential query process

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Oldies but Goodies COMSOC Video Seminar
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TU games and simple games

- A monotone TU-game is a cooperative game $G = (X, v)$ given in coalitional form: for the player set X there is a map $v : 2^X \to \mathbb{R}$ with $v(\emptyset) = 0$, and $A \subseteq B$ implies $v(A) \leq v(B)$.
- A simple game is a TU game where $v(A) \in \{0, 1\}$ for all A.
- A winning coalition in a simple game is a subset $A \subseteq X$ with $v(A) = 1$.
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$$\partial_i v(A) = v(A) - v(A \setminus \{i\}).$$

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Sequential random query process for simple games

- Sample uniformly without replacement until we have seen enough players to have a winning coalition.
- The number Q of queries is at least $k+1$ if and only if the first k do not form a winning coalition.
- Hence Q is a random variable with expectation

$$E[Q(G)] = n+1-\sum_{k=1}^{n} P(\text{random coalition of size } k \text{ is winning}).$$

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Motivating example

- Devise a less crude measure of the manipulability of a voting rule.
- The most commonly used measure simply gives 0 or 1 for a given profile, depending on whether it is manipulable by some coalition or not.
- A more sophisticated measure is the size of the minimum manipulating coalition.
- We want to see how easy it would be to assemble a manipulating coalition - the query model is one idea.
- Concrete example: Borda rule, 2 voters abc, 1 voter bac, 1 voter cba.

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Collective and individual measures

- The process is equivalently described via first choosing a probability measure μ_n on $\{0, \ldots, n\}$, sampling from it and then choosing a subset of that size uniformly at random.
- The function $Q_F^*(G)$ is a measure of decisiveness that generalizes Coleman's index.
- The marginal $q_F^*(G)$ of $Q_F^*(G)$ is a power index.
- The simplest F (corresponding to uniformly choosing the size of the subset) gives a new measure, that we call $Q_0^*(G)$.
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Generalization to TU-games

- The obvious definition is

$$Q_F^*(G) = \sum_{S \subseteq X} \binom{n}{|S|} \mu_n(|S|) v(S)$$

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Name	$F(n,k)$	$\mu_n(k)$	Value on voting game
Coleman/Banzhaf	$2^{-n}\sum_{j\geq k}\binom{n}{j}$	Bin$(1/2)$	$3/4 = 0.75$
Shapley	$1 - H_{k-1}/H_n$	Unif$(1..n)$	$36/50 = 0.72$
Q_0^*/q_0^*	$1 - k/(n+1)$	Unif$(0..n)$	$2/3 \approx 0.67$

Another connection to the Shapley value

- In applications of Shapley the grand coalition is always winning and we seek to divide up the surplus.
- In some applications the grand coalition is not winning (e.g. the voting game example above).
- If we allow this, our q_F^* are precisely analogous to semivalues and the characterization theorem of Dubey, Neyman & Weber (1981) extends naturally.
- In this new model, q_0^* is the exact analog of the Shapley value.

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Future work

- An entire class of values/power indices and potentials/decisiveness indices remains to be explored.
- Axiomatic characterization of these new values.
- Finding applications of new values such as q_0^*. We give applications to manipulability of voting rules. Perhaps machine learning applications?
- Q_0^* appears better than Coleman in discriminating between simple games, and should be studied more.

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