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Models of inter-election change in partisan vote share

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Abstract

For a two-party electoral competition in a districted legislature, the change in mean vote share for party A from one election to the next is commonly referred to as swing, though this term also has other meanings in the election literature. A key question, highly relevant to election forecasting and the measurement of partisan gerrymandering, is: "How do we expect the swing to be distributed across the districts as a function of previous vote share (and perhaps, other factors)?".

The literature gives two main answers to that question: uniform swing and proportional swing. Which is better has been unresolved for decades. Here we (a) provide an axiomatic foundation for desirable properties of a model of swing; (b) use those axioms to demonstrate why using uniform swing or proportional swing is a bad idea, (c) provide a reasonably simple swing model that does satisfy the axioms, and (d) show how to integrate a reversion to the mean effect into models of inter-election swing.

We show that all the above models can be expected to work well when (a) elections are close, or (b) when we restrict ourselves to data where swing is low, or (c) when we eliminate the cases where the model is most likely to go wrong. In particular, sometimes the model tested in the literature is not the standard model, in that either (c1) a piecewise or truncated variant of the model is being used, or (c2) there is an arbitrarily chosen correction (usually 75%) for districts that are uncontested. As we show empirically with data from U.S. congressional elections, the choice of such correction makes a substantial difference to results. We also show that in addition to its superior axiomatic properties, our new model provides an overall equal or better fit on five indicators: mistakes about directionality of change, mistakes in winner, estimates that are outside the [0..1] bounds, mean-square error, and correlation between actual and predicted values. We recommend replacing the uniform and proportional swing models with the new model in almost all applications.

Keywords

uniform swing, proportional swing, vote share prediction

1 Introduction

There are many situations where we have data at two (or possibly more than two) points in time, and we are interested in comparing values between and among the data points for each subject or case. One such situation is an experiment in which there is a treatment effect which results in an overall mean change of s units in some variable of interest after the intervention. The situation in which we are chiefly interested here is a two-party plurality vote contest (parties A and B) in the set of electoral units (districts, states, etc), for which we have data at two distinct points in time. For simplicity, we assume an idealized situation in which there are K districts each of equal size, and turnout is equal in each district.

Consider two elections, one at time $t = 1$ and one at time $t = 2$. Let x_i denote the vote share of a given party at time 1 in district i, and x'_i the vote share at time 2 in that district. We use bars to denote mean values over districts, so that $\overline{x}$ denotes the mean over all districts of x_i, namely the overall vote fraction for that party.

The **aggregrate inter-election swing** [1] is simply $\overline{x}' - \overline{x}$, and we denote this by s. By symmetry, in a two party contest, swing for party A is swing against party B, and conversely. At the district level we denote by s_i the district-level inter-election district swing $x'_i - x_i$.

A key problem in election forecasting and the study of partisan gerrymandering, among other areas, is simply: given s and the previous election result, estimate s_i for each i. The literature gives two main answers to the above question: **uniform swing** and **proportional swing**.

Butler, in a chapter in McCallum and Readman (1999, pp. 263–265) is credited with first using uniform swing to model British elections. While its limitations for multi-party contexts were noted by Butler and by later authors such as Curtice and Steed (1982); Rose (1991); Dorling et al. (1993), the concept has nonetheless subsequently become a workhorse model in the U.K. Multiparty competition in the U.K. is often dealt with by focusing on competition between

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the two largest parties. The use of the uniform swing model was promoted by Butler and Van Beek (1990) and is now common in the U.S., with a similar way of treating multi-party competition. Following Gelman and King (1994), swing is sometimes taken to be uniformly distributed with a stochastic error term with mean zero and some fixed variance (often estimated from data on the standard deviation of past inter-election vote share shifts in the polity). Some stochastic versions of swing include covariates, such as incumbency, in the model (see e.g. Katz et al. (2020)). When we study congressional elections in the U.S. we also restrict ourselves to two-party competition.

Proportional swing, on the other hand, posits that each unit experiences the same percentage change from one election to the next, and thus units that have higher starting values experience greater absolute vote changes. We believe that this model was first proposed by Berrington (1965), where it was offered as an alternative to uniform swing. In subsequent literature, proportional swing has become the main alternative to uniform swing.

Even in the context of two-party politics the use of these models has always been problematic, since each had properties that made a good fit to empirical data implausible. And yet, when applied to British elections, by Butler and then by other authors in every post-WWII British National Election Study, or in U.S. congressional and other elections, the fit of these models, especially uniform swing, has been shown to be amazingly good – see e.g. Butler and Stokes (1969, pp. 140-151) and Katz et al. (2020). The remarkable empirical fit of uniform swing led Butler and Stokes (1969) to talk about the "paradox of swing". They suggested that, on theoretical grounds, proportional swing should work better. Work of Johnston (1981, 1983) showed that uniform swing did not perform perfectly in all situations and any explanation for its effectiveness must be rather complex. However proportional swing did not win out. As Gershuny (1974, p. 116) puts it, after arguing for fundamental flaws with the proportional swing model: "The mass of the evidence is however that swings are virtually equal across constituencies." And this is a finding that has been confirmed by subsequent decades of observations.

We find neither swing model to be satisfactory. While many authors have criticized the uniform swing and proportional swing models, to our knowledge no author has systematically sought to enumerate the desirable properties of a model of inter-election swing. In Section 2, we consider a general deterministic model of inter-election swing, of the form

$$s_i = f(x_i, s, \overline{x}_i)$$

where x_i is the vote share of party A in the ith district at time 1. We provide an axiomatic characterization of the properties we wish the function f to satisfy.

Having identified three key axioms, we next turn to evaluating uniform swing and proportional swing with respect to the axioms, discovering as we should have expected, that neither of these models is fully satisfactory. In Section 2.2, we show that uniform swing violates one of the three conditions, and proportional swing violates two of them.

We next look to see whether there are any functions that do satisfy all the axioms. After proving an impossibility result applicable to a large family of models including both uniform and proportional swing, we find a positive solution in Section 2.2.3, via a piecewise defined function that we feel has strong substantive arguments for adoption, and which has the simplest functional form we can find.

We also show how to incorporate a reversion (sometimes called "regression") to the mean effect while still satisfying our first three axioms. In its simplest form reversion to the mean posits that outcomes are a product of fundamental factors and idiosyncratic factors some of which will be constituency-specific. Idiosyncratic features can be represented as stochastic errors occurring with mean zero. Since errors are posited to be uncorrelated across elections, the implication of a reversion to the mean effect is that on average, outcomes will reflect fundamentals. But then especially large victories (defeats) are likely to involve idiosyncratic features which cannot be expected to persist into the next election. In the concluding discussion we will consider factors other than a reversion to the mean effect that might be added to more realistically model inter-election swing.

In Section 3 we consider how our axiomatic results relate to empirical analyses, beginning with "toy" results to illuminate the properties of the various models and culminating in analyses of two-party vote share in tends of thousands of pairs of adjacent U.S. congressional elections from 1968-2016. In Section 3.1 we demonstrate with hypothetical data that, for close districts and small swings, all models will give very similar predictions. In Section 3.2 we demonstrate with actual data that we can improve an already very good fit of these two standard models even further by changing the structure of the data, either by entirely eliminating certain extreme cases (uncompetitive districts) or by recoding uncompetitive districts from 100% to 75%.

Our modified model provides an overall equal or better fit to U.S. congressional data than the traditional models, while having much nicer axiomatic properties.

Our results aid us in making sense of the perceived good fit of models such as uniform swing, especially once we recognize that some applications of these models are not actually uses of the standard model. For example, when Katz, King and Rosenblatt (2020) assess the fit of uniform swing and proportional swing they are assigning non-competitive districts to be 75%-25% districts. In other studies of U.S. elections, such as ones to calculate seats-votes relationships, it is very common to see uncompetitive districts coded as 75% districts (or the replacement of results in an uncontested election for the constituency with data from a more competitive constituency wide election projected into that district) in order to avoid what would otherwise be seen to be misleading out of bounds results. Practices such as this recoding are taken for granted as minor. For example, Katz et al. (2020, endnote 6, p171) view this as "standard practice." They also observe that the effects of this imputation "have no material impact on our results." While generally correct, this is too strong a claim, as we discuss below. We are preparing a companion article with a more empirical focus, which in particular will study how choices in how one measures goodness of fit can play a large role in whether a given swing model gives convincing results.

2 Axiomatic properties for functions modelling inter-election swing

Recall our basic idealized situation. There are K districts of equal size and two parties, A and B, contesting all districts. Unless otherwise specified we state results for party A, whose vote share is denoted x_i.

We use the following toy example throughout, in order to concretely illustrate ideas.

Example 2.1. (Toy example) There are two parties and two districts of equal size. Fix a parameter α with $0 \leq \alpha \leq 1/2$. In District 1, party A has fraction $1-\alpha$ of the votes, and party B receives α, while these vote shares are reversed in District 2. The overall vote share of each party is $(\alpha + (1-\alpha))/2 = 0.5$, because the districts have the same size.

Definition 2.2. The **district-level swing** in district i is given by

$$s_i := x'_i - x_i.$$

The **aggregate swing** is given by

$$s := \overline{x}' - \overline{x}.$$

Note that since $\overline{x}' = \overline{x} + s$, we must have $0 \leq \overline{x} + s \leq 1$ and so $-\overline{x} \leq s \leq 1 - \overline{x}$.

Definition 2.3. By a *naive swing model* we mean a prediction of x' of the form

$$x'_i = x_i + f(x_i, s)$$

where $f \equiv f_A$ is a fixed function (depending only on A but not i or s).

Alternatively, $s_i = f(x_i, s)$ for all i. Note that the district-level swing depends on the previous vote fraction in that district, but not on the votes in other districts except via the overall swing.

There are some obvious requirements for such a predictive model. The definition of district-level and aggregate swing yields

$$\frac{1}{K}\sum_{i=1}^{K} f(x_i, s) = s \qquad \textbf{(mean swing condition)}. \qquad \text{(a1)}$$

The next condition holds because x'_i is bounded within the interval $[0, 1]$:

$$0 \leq x_i + f(x_i, s) \leq 1 \qquad \textbf{(respecting bounds)}. \qquad \text{(a2)}$$

We also require a third condition, which we may think of as a symmetry condition similar to neutrality in social choice theory, namely that a universal swing model for two-party competition must give the same answer whether we look at either party. This translates to

$$f(x_i, s) + f(1 - x_i, -s) = 0 \qquad \textbf{(neutrality)}. \qquad \text{(a3)}$$

If we think of the vote shares of the parties as arranged in a matrix with one row per party and one column per district, (a1) can be interpreted as a row sum condition, and (a3) as a column sum condition.

2.1 Existing models of inter-election swing

As mentioned above, the **Uniform Swing** model is given by

$$f(x_i, s) = s \qquad \text{for all } i.$$

Thus $x'_i = x_i + s$ for all i. In other words, the same net fraction of voters changes to party i in each district. For example, in the toy example with $\alpha = 0.2$, a 6% swing to A changes its vote in District 1 to 86% and in District 2 to 26%.

Although it is easily seen that Uniform Swing satisfies (a1) and (a3), it does not satisfy (a2). For example, in the toy model, the upper bound for party A in District 1 is violated when $\alpha > 1/2$ and $s > 1 - \alpha$. Note that the lower bound for B in District 1 would also be violated.

There is a good reason why we have not presented a naive swing model satisfying all axioms (a1) – (a3): namely that no such model exists.

Proposition 2.4. *No naive swing model can satisfy both* (a1) *and* (a2).

Proof. Consider the situation where all district vote shares are equal, so that $x_i = \overline{x}$ for all i. Then (a1) implies that $f(\overline{x}, s) = s$ for all s. Since $\overline{x}$ can take any value in the interval $[0, 1]$, this shows that $f(x_i, s) = s$ for all s, in general. In other words, the row sum condition is enough to yield the uniform swing model, and since the uniform swing model fails (a2), the claimed impossibility follows.

Thus in order to obtain a broader class of well-behaved simple models, we need a more general functional form for f.

Definition 2.5. A **swing model** is a prediction of the form

$$x'_i = x_i + f(x_i, s, \overline{x})$$

for some function f.

The properties (a1) – (a3) above correspond to analogous properties, with the same motivations:

$$\frac{1}{K}\sum_{i=1}^{K} f(x_i, s, \overline{x}) = s \qquad \text{(A1)}$$

$$0 \leq x_i + f(x_i, s, \overline{x}) \leq 1 \qquad \text{(A2)}$$

$$f(x_i, s, \overline{x}) + f(1 - x_i, -s, 1 - \overline{x}) = 0 \qquad \text{(A3)}$$

In our view these three conditions are minimal requirements, essential for any logically consistent theory of swing. The first follows directly from the very definition of district-level and overall swings. The second guarantees that a prediction has the correct form and is not logically ruled out from being correct. The third seems clear because any universal theory should surely not depend on the specific parties involved, or which of them we focus on at the moment.

The only other prominent swing model in the literature is **Proportional Swing**, defined via the formula

$$f(x_i, s, \overline{x}) = s x_i / \overline{x}.$$

Note that the definition implies that

$$\frac{x'_i}{\overline{x}'} = \frac{x_i}{\overline{x}}$$

so that the relative share among districts of the party's overall vote does not change between elections. For example, in the toy model with $\alpha = 0.8$, a 6% swing to A is a 12% relative increase in its overall vote share. The Proportional Swing prediction in District 1 is 89.6% and in District 2 it is 22.4%. Of course if we think of this as a -6% swing to B, then this is a relative decrease of 12% in the overall vote share of B, and hence the vote fraction of B in the first district under this model should be 17.6% and in the second district 70.4%. This causes a violation of the bounds condition (A2) and also shows that (A3) fails. This deficiency of Proportional Swing has been noticed early, for example McLean (1973); Gershuny (1974).

It is easily verified that Proportional Swing does satisfy (A1).

2.2 New models

Beginning with uniform swing and proportional swing models, a natural question is whether there are small changes that can be made to one or both of those models that would satisfy axioms (A1)–(A3). We now explore this, and some false starts we eventually find one, in Section 2.2.3.

Do there exist any models that satisfy all axioms (A1)–(A3)? The answer is yes, since there are so many degrees of freedom. Consider the special case with 2 districts. Then we seek a mapping from a 2×2 contingency table with column sums equal to 1 and fixed row sums r_1, r_2 to another such table with r_1, r_2 replaced by r'_1, r'_2, subject to $r_1 + r_2 + s = r'_1 + r'_2$. We have already accounted for (A1) and (A3). To satisfy (A2) we require that given a vote share a for party A in district 1, we can choose vote share a' for the same party in the same district such that $0 \leq a' \leq 1$ and $0 \leq a + b + 2s - a' \leq 1$. This is always possible because of the constraints on s. Thus it is indeed possible to satisfy all three axioms. The question is *whether we can do so with a reasonably simple and understandable functional form*.

We now look for a model that satisfies at least axioms (A1)–(A3), by considering some models defined by simple formulae.

2.2.1 Truncations One variant of the uniform swing model would be simply to impose truncation, i.e., to set $x_i + f(x_i, s) = 1$ if $x_i + s \geq 1$ and $x_i + f(x_i, s) = 0$ if $x_i + s \leq 0$. Note that this procedure of simply truncating out of bounds values to be either 0 or 1 was used by expert witnesses estimating racially polarized voting in voting rights cases using Goodman's (Goodman 1953, 1959) method of ecological regression. However, that methodology has largely been replaced with ecological inference techniques (King, 1997) that assure that estimates are within bounds. In any case, in our situation the truncation procedure leads to failure of axiom (A1).

2.2.2 Models that are linear in s We start with the family of linear models given by formulae of the type

$$s_i = sg(x_i, \overline{x}).$$

Of course, we mean linear in the variable s — the functional dependence on x is not specified. If $g(x_i, \overline{x}) = 1$, this gives Uniform Swing. The Proportional Swing model is also in this family, with $g(x_i) = x_i/\overline{x}$.

Example 2.6. McLean (1973) considers models using transition matrices, in which we account for the fraction of voters of party i that vote for party j in the next election (this includes abstainers as a party). Consider the simple case of no abstainers and two parties. If a is the fraction who switch from party 1 to party 2, then in order to satisfy row and column sums, the matrix must have the form $\left(\begin{smallmatrix} a & 1-a \\ 1-a & a \end{smallmatrix}\right)$. This leads after some algebra to

$$x'_i = x_i + s(1-a)(1-2x_i).$$

In other words, this is a model in the same family with $g(x) = (1-a)(1-2x)$ (swings go against the national one in districts dominated by party 1, but toward it in districts in which it is weak). Such a model violates the bounds when $x_i = 1$ or $x_i = 0$ if the swing is against party i. Furthermore its behaviour is counterintuitive in that the swing is zero precisely for elections that are the most competitive.

Such a model satisfies (A1) if and only if C is the reciprocal of the average of $g(x_i)$ over all districts: $C = 1/\overline{g(x)}$. A sufficient condition for symmetry is that $g(x) = g(1-x)$, so that the function g is symmetric about $x = 1/2$. In order to satisfy the bounds condition (A2), we need $g(0) = g(1) = 0$, so that swing is zero in completely lopsided districts. For example, we could use a quadratic function $g(x) = x(1-x)$.

So far we are proceeding well, but our search is in fact pointless, a result that surprised the authors.

Proposition 2.7. *No swing model linear in s can satisfy both (A1) and (A2).*

Proof. We recall the toy model where $1/2 > \alpha > 0$. Consider the effect in its strong district of a swing of $1/2$ to A. By (A2), we must have

$$1 - \alpha + \frac{1}{2} g(1-\alpha, 1/2) \leq 1$$

so that

$$g(1-\alpha, 1/2) \leq 2\alpha.$$

By a similar argument, this time using the lower bound and considering the effect in the other district of a swing of -2α, we obtain

$$g(\alpha, 1/2) \leq 2\alpha.$$

However condition (A1) then implies that $2 = g(1-\alpha, 1/2) + g(\alpha, 1/2) \leq 4\alpha < 2$, a contradiction.

2.2.3 Piecewise linear model In view of the negative results above, we differentiate between positive and negative swings. Our experience with the models seen so far leads after some educated guessing to the following:

$$f(x_i, s, \overline{x}) = \begin{cases} s\frac{1-x_i}{1-\overline{x}} & \text{if } s \geq 0; \\ s\frac{x_i}{\overline{x}} & \text{if } s < 0. \end{cases}$$

Straightforward algebra shows that this model satisfies (A1), (A2) and (A3). This is the simplest functional form we can find for which (A1)–(A3) are satisfied.

Note that under this model (as with the previous two), the sign of the district-level swing is the same as the sign of the overall swing. Under positive swings, in districts in which a party is relatively strong (its vote share is more than its overall average) the district-level swing is smaller than in districts in which the party is relatively weak. However for negative swings, an overall swing against the party is amplified in its stronger districts and diminished in its weaker districts.

A substantive argument for this model is as follows. If in each district there are swing voters as well as partisans of each party, along lines similar to the argument in McLean (1973), we may readily imagine that, in districts where A already scores highly, there are relatively few swing voters left to convince, so that an overall swing toward A does not improve that party's vote share substantially in such districts. However in districts where A scored relatively low, there is more chance of winning over swing voters. However if the swing is away from A, the reverse is true (alternatively, the same is true of B).

Table 1. Axioms satisfied by swing model

Model/Axiom	A1	A2	A3
uniform	✓	✗	✓
proportional	✓	✗	✗
truncated uniform	✗	✓	✓
linear in s	✓	✗	(✓)
piecewise	✓	✓	✓

We note in passing (a fact discovered after the present article was written) that what Katz et al. (2020) call proportional swing in their article is revealed by inspection of their R code to be exactly the piecewise model [2].

2.2.4 Models incorporating a mean-reverting term

Another condition that is satisfied by all the models we have discussed so far is:

$$f(x_i, 0, \overline{x}) = 0. \tag{A4}$$

This condition says that when there is no aggregate swing, our best predictor of outcomes in individual districts is that they will be unchanged from the previous period. We do not regard this as a desirable property to impose generally on inter-election swing. The reason is quite simple: (A4) is inconsistent with observation, in that extremely high or low vote shares often change between elections even with no aggregate swing.

Suppose that we add a term $h(x_i, \overline{x})$ to any of the above formulae already satisfying all three axioms. Then (A4) no longer holds, (A1) holds if and only if h has mean zero, while (A3) holds if and only if $h(x_i, \overline{x}) + h(1 - x_i, 1 - \overline{x}) = 0$. The bounds condition would be satisfied provided $h(x_i, \overline{x}) \geq 0$ when $x \leq \overline{x}$ and $h(x_i, \overline{x}) \leq 0$ when when $x \geq \overline{x}$.

The simplest functional form satisfying the above conditions on h is $h(x_i, \overline{x}) = c\,(\overline{x} - x_i)$, for some constant $c > 0$. Note that this model implies that the magnitude of inter-election shift is affected (in a linear way) by how far away the previous outcome was from the average outcome. Also note that the model has one free parameter, unlike the previously considered models.

We note in passing that adding a mean-reverting term to a model that does satisfy the other axioms, but not the bound condition (A2), can also work. For example, define a model to be *affine in s* if it has the form

$$f(x_i, s, \overline{x}) = sg(x_i, \overline{x}) + h(x_i, \overline{x})$$

for some function h, so we have added a term independent of s to a model that is linear in s. Thus every linear model is an affine model with $h = 0$. We would like to prove an impossibility result similar to Proposition 2.7 for affine models, but such a result is false.

Example 2.8. The affine model with $g(x_i, \overline{x}) = 1$ and $h(x_i, \overline{x}) = \overline{x} - x_i$ satisfies (A1), (A2) and (A3). Note that $x'_i = \overline{x'}$, so that this model is rather trivial, in that it predicts the same vote share in each district, independent of the previous results.

We have not found an affine model with dependence on i that satisfies our three axioms, but neither have we proved it to be impossible.

3 Behavior of models

In this section, we investigate the behaviour of our models on some specific vote distributions.

3.1 Artificial data

Suppose that we have complete uniformity across districts: in every district, $x_i = \overline{x}$. Then, as expected, all the models discussed above make the same prediction, namely $x'_i = x_i + s$ for each i.

In Table 2 we consider a very polarized version of the toy model, with $0 \leq \alpha \leq 1/4$. Note that there are measurable differences between the predictions made. For example, an initial vote share for party A of $(95\%, 5\%)$ in District 1 and District 2, combined with a swing to party A of 10%, yields $(105\%, 15\%)$ for the uniform model and $(99\%, 21\%)$ for the piecewise model.

In Table 3 we consider the toy model with two very competitive districts, where $\alpha = 1/2 + \varepsilon$ and $0 \leq \varepsilon \leq 1/4$. Note that the predictions of all models are the same to first order in ε, and so unlikely to be distinguishable for fairly small ε. For example, an initial vote share for party A of $(55\%, 45\%)$ in District 1 and District 2, combined with a swing to party A of 10%, yields $(65\%, 55\%)$ for the uniform model, $(66\%, 54\%)$ for the proportional model, and $(64\%, 56\%)$ for the piecewise model.

So far we tentatively conclude that for close districts and small swings, all models will give very similar predictions. On lopsided districts, they will not agree in their vote share prediction, but the predicted winner will be the same in most cases unless the swing is very large.

3.2 Real data

We considered real data taken from the dataset Katz et al. (2019) used in Katz et al. (2020), consisting of district-level data on Democratic and Republican vote share for elections in United States state legislatures over the period 1968-2016, containing over 140000 data points. The basic unit of analysis is an election in a fixed district in a fixed state, at two successive elections between which redistricting has not occurred. This gave just over 73000 units. Uncontested

Table 2. Predictions for "polarized" example with 2 parties and 2 districts, swing of 2α to A

model / (party, district)	(A,1)	(A,2)	(B,1)	(B, 2)
original	$1-\alpha$	α	α	$1-\alpha$
uniform	$1+\alpha$	3α	$-\alpha$	$1-3\alpha$
proportional	$1+3\alpha-4\alpha^2$	$\alpha+4\alpha^2$	$\alpha-4\alpha^2$	$1-5\alpha+4\alpha^2$
piecewise	$1-\alpha+4\alpha^2$	$5\alpha-4\alpha^2$	$\alpha-4\alpha^2$	$1-5\alpha+4\alpha^2$

Table 3. Predictions for "competitive" example with 2 parties and 2 districts, swing of 2ε to A

model / (party, district)	(A,1)	(A,2)	(B,1)	(B, 2)
original	$1/2-\varepsilon$	$1/2+\varepsilon$	$1/2+\varepsilon$	$1/2-\varepsilon$
uniform	$1/2+\varepsilon$	$1/2+3\varepsilon$	$1/2-\varepsilon$	$1/2-3\varepsilon$
proportional	$1/2+\varepsilon-4\varepsilon^2$	$1/2+3\varepsilon+4\varepsilon^2$	$1/2-\varepsilon+4\varepsilon^2$	$1/2-3\varepsilon-4\varepsilon^2$
piecewise	$1/2+\varepsilon+4\varepsilon^2$	$1/2+3\varepsilon-4\varepsilon^2$	$1/2-\varepsilon-4\varepsilon^2$	$1/2-3\varepsilon+4\varepsilon^2$

elections are a prominent feature of US elections, and we need to deal with that issue. We made 3 versions of the dataset: uncontested elections count as 0.75 vote share to the winner and 0.25 to the other party; uncontested elections count vote share 1.0 to the winner; all uncontested elections are removed from the dataset. We adapted code from Katz et al. (2019); all our code is available at Harvard Dataverse Wilson (2021). [3]

We compare performance of swing models as follows. Given as input the entire vote counts for all parties and districts for Election 1, and the popular vote counts from Election 2 (from which we can compute the swing to/from each party), we compute the prediction of each model for Election 2 for each party in each district. We measure quality of prediction in five ways: the fraction of units in which the winner is correctly predicted; the fraction of times when the district-level sign is correctly predicted the fraction of units in which the vote share prediction stays in bounds; the the mean squared error in vote share prediction; the Pearson correlation between the real and predicted vote share data. Note that all three models predict the sign of each district-level swing to be the same as the sign of the aggregate swing. Thus all should give the same result for sign in most cases, but in the case where the original vote share is 100% (dataset unc1.0) and there is positive aggregate swing, the uniform and proportional will predict +1 for sign whereas the piecewise will predict 0. If the following election is also uncontested, the piecewise model outperforms the others.

Results of our tests are displayed in Table 4, displayed to 3 significant figures. Bold entries indicate the best performance among the models on the given measure corresponding to the column for the given dataset. Note that the piecewise measure always has perfect performance on the bounds measure. We show the results for the first and fourth quartile in Tables 5 and 6. Table 7 describes the behavior on close races (winner scores in the range [50%, 52.5%]).

First, the results show that the convention we choose for treating uncontested elections has a large influence on the ability of the methods to predict the winner or the sign of the district-level swings, as well as the correlation.

Second, all models had difficulty in predicting the sign of district-level swings, scoring around 50% or even substantially lower, worse than flipping a coin, in some scenarios. They all performed much better on the other measures. Of course, piecewise swing never violates the bounds, so it scores perfectly on this measure. However the other models also scored very highly on this measure except when uncontested elections are given vote share 1 for the winner, and even then they scored over 80%. The correlation scores were all over 80%.

Third, as expected, for small swings the predictions of all three models are almost indistinguishable, except when uncontested elections are given vote share 1 for the winner, when uniform swing performs relatively poorly on the sign and bounds measures. For the uniform and piecewise models the probability of predicting the winner was around 93%, 89%, and 85% under the three conventions, and in each case the two probabilities differ by less than 0.2%. The similarity was even greater in the correlation, while the mean-square error had somewhat larger differences but still of the order of 1-2%,

Fourth, there was a consistent very small advantage to uniform swing in predicting the winner, and to the piecewise model in predicting sign, mean-square error and correlation.

Drilling deeper into the data, we considered the subsets of the three datasets for which the aggregate swing is in the first quartile ("elections with small swing"), or in the fourth quartile ("elections with large swing"). We also looked at elections whose *variance* in district-level swing was in the first quartile as opposed to those in the fourth quartile.

The relative performance of methods was almost identical those in the overall dataset, with a tiny advantage to uniform swing in predicting the winner, but the piecewise model leading on the other four indicators. All methods found it noticeably easier to predict the winner, and harder to predict the sign of the district-level swing, when dealing with small swings, and the reverse was true for large swings.

Finally, we considered the performance of the models on close races, which we defined to be those pairs of contested elections where the winner scored between 50% and 52.5% in the second election. All measures performed relatively poorly in this case at predicting the winner, scoring only around 57% — as expected, close races are harder to call. Although they did better on predicting the sign of the district-level swing, the correlation dropped to just over 20%. However, the piecewise model did at least as well as the others on every measure.

Table 4. Results for swing models on dataset from Katz et al. (2020). Bold entries indicate the best performance among the models on the given measure corresponding to the column for the given dataset.

dataset	model / measure	winner	sign	bounds	mean-square	ρ
unc0.75	uniform	0.932	**0.497**	**1.000**	0.00747	0.903
unc0.75	proportional	**0.933**	**0.497**	0.999	0.00756	0.902
unc0.75	piecewise	0.930	**0.497**	**1.000**	**0.00728**	**0.903**
unc1.0	uniform	**0.904**	0.498	0.832	0.0381	0.817
unc1.0	proportional	0.904	0.539	0.884	0.0389	0.813
unc1.0	piecewise	0.892	**0.604**	**1.000**	**0.0360**	**0.818**
cont only	uniform	**0.855**	**0.678**	**1.000**	0.00521	**0.891**
cont only	proportional	0.853	**0.678**	0.999	0.00533	0.889
cont only	piecewise	0.852	**0.678**	**1.000**	**0.00509**	0.891

Table 5. Results for swing models on dataset from Katz et al. (2020), first quartile mean swing. Bold entries indicate the best performance among the models on the given measure corresponding to the column for the given dataset.

dataset	model / measure	winner	sign	bounds	mean-square	ρ
unc0.75	uniform	**0.951**	**0.385**	**1.000**	0.00734	**0.909**
unc0.75	proportional	**0.951**	**0.385**	**1.000**	0.00735	**0.909**
unc0.75	piecewise	**0.951**	**0.385**	**1.000**	**0.00730**	**0.909**
unc1.0	uniform	**0.951**	0.411	0.835	0.0367	**0.826**
unc1.0	proportional	**0.951**	0.463	0.873	0.0368	**0.826**
unc1.0	piecewise	**0.951**	**0.517**	**1.000**	**0.0362**	**0.826**
cont only	uniform	**0.886**	**0.528**	**1.000**	0.00548	0.882
cont only	proportional	**0.886**	**0.528**	**1.000**	0.00549	0.882
cont only	piecewise	**0.886**	**0.528**	**1.000**	**0.00544**	**0.883**

Table 6. Results for swing models on dataset from Katz et al. (2020), fourth quartile mean swing. Bold entries indicate the best performance among the models on the given measure corresponding to the column for the given dataset.

dataset	model / measure	winner	sign	bounds	mean-square	ρ
unc0.75	uniform	**0.894**	**0.672**	**1.000**	0.00734	**0.895**
unc0.75	proportional	0.890	**0.672**	0.998	0.00754	0.891
unc0.75	piecewise	0.884	**0.672**	**1.000**	**0.00698**	**0.895**
unc1.0	uniform	**0.857**	0.604	0.840	0.0414	0.802
unc1.0	proportional	0.854	0.642	0.894	0.0426	0.792
unc1.0	piecewise	0.838	**0.705**	**1.000**	**0.0372**	**0.805**
cont only	uniform	**0.816**	**0.842**	**1.000**	0.00509	0.887
cont only	proportional	0.811	**0.842**	0.996	0.00530	0.882
cont only	piecewise	0.806	**0.842**	**1.000**	**0.00489**	**0.888**

Table 7. Results for swing models on dataset from Katz et al. (2020), close races. Bold entries indicate the best performance among the models on the given measure corresponding to the column for the given dataset.

dataset	model / measure	winner	sign	bounds	mean-square	ρ
cont only	uniform	0.576	**0.734**	**1.000**	0.00429	0.212
cont only	proportional	0.576	**0.734**	**1.000**	0.00432	0.211
cont only	piecewise	**0.581**	**0.734**	**1.000**	**0.00368**	**0.218**

4 Discussion and conclusion

Why axioms?

Our approach here is influenced by that of Taagepera (1999, 2007) where great attention is paid to the functional form of relationships and how these are affected by boundary conditions that rule out certain outcomes as logically impossible. Taagepera (2008) argues that political science research is overly wedded to statistical models and to linear models, and that this unbalances and limits the field. Searching for predictive models that respect logical constraints and are grounded in substantive theories of political behavior is not only more likely to yield useful models, but spurs investigation of fundamental principles.

Shugart and Taagepera (2017) lay out theoretical reasons why we might expect observed mean values of some key election parameters (e.g., effective number of political parties) to be a function of the geometric mean of worst case and best case scenarios. Of course, for any specific county in any given election the actual results will not match the mean *a priori* expectation, but it is still useful to derive axiomatic expectations about the likely mean outcomes for a set of countries, and to test to see how well those expectations are fulfilled. Their derivation of interlocking models for distribution of seats, votes, numbers of parties, etc, in the context of simple electoral systems, is a good example of the kind of result we seek by setting out on the path of axiomatics.

In the context of electoral swing, even if a purely statistical or machine learning approach could discover a good fit to existing data, our confidence in its performance on necessarily out-of-sample, future data should not be high. The total number of political elections in human history is not large compared to the enormous corpuses typical in areas considered to be machine learning success stories.

Our theoretical analysis showed that the piecewise model has much better axiomatic properties than the better known uniform and proportional models. Our toy models suggest strongly that the piecewise model will give results very close to the uniform and proportional swing models for "normal" elections, but more accurate predictions for districts where the previous result was not close and/or swings are large.

Empirical issues

On the dataset we analysed, uniform and piecewise models appear to perform similarly, each scoring slightly better than the proportional model on most criteria, and the differences in the model predictions are very small in most cases.

Interestingly, the piecewise model is slightly worse than the others at predicting the winner in many cases, but has better ability to predict the sign of district-level swings, generally lower mean square error on each dataset, and better correlation than the other models. It also performs better on close races, although all models perform fairly poorly in that case. We reemphasize that the choice made in how to deal with uncontested elections is very important.

The uniform swing model has been considered for decades to give a "good enough" fit to data, despite having weak theoretical foundations. It is very hard in many cases to distinguish predictions made using uniform swing from those using the theoretically superior piecewise model. For readers with a practical inclination, this should bolster their belief that uniform swing works well in practice. However, we advise caution for several reasons. First, none of the models here works particularly well in predicting results of close elections or when swings are large. If we are only interested in the winner, swings are small and the previous election is not close, it is easy to guess who will win this time. Second, as mentioned above, the total number of political elections held in the history of humankind is still rather small, and so any conclusion based on real data should be treated cautiously. It is easy for any model to fit well on particular datasets. For example, over a limited range of values a linear approximation can fit very well a non-linear curve, but extrapolating that linear function will yield very erroneous estimates. Third, swing models are used not only for election forecasting but also for applications such as the study of counterfactual elections, for example when considering partisan gerrymandering or electoral system design more generally, and these may take us outside the realm of small swings. Fourth, our results here may have wider application beyond the electoral context, for example to experiments whose treatment condition may have variable effects on subjects. In those situations the still-mysterious confluence of factors that cause uniform swing to work well in many political contexts may not eventuate at all.

Future work

One way to relax our constraints is to consider probabilistic models (all models in the current paper are deterministic "mean-field approximations"), as in Note 2 below. We expect that for probabilistic models, axioms similar to (A1) – (A3) will be needed, but for means of random variables rather than deterministic values. In the rest of this section we restrict to the deterministic case, which is the main focus of this paper. We expect that improvements in deterministic models will yield substantial insight into improving stochastic ones.

Clearly, we have not explored the entire space of swing models satisfying our three axioms. There is still plenty of room for exploration of further swing models, which balance predictive accuracy with functional simplicity, substantive explanations and good axiomatic properties. Although the three models we focused on here have similar performance, none of them is very good at predicting close elections, and there is surely a reasonable chance of finding a model with better performance. In order to do this, it may be necessary to move beyond models satisfying (A4), for example by adding a reversion to the mean effect as described above.

Further, while we have shown how to incorporate a regression to the mean effect, there are other potentially important factors we have not incorporated. In particular, we assume that swing moves in the same way in each of the districts as it does for the polity as a whole. But that assumption is violated if, for example, there is a simultaneous realigning trend that works differently in different parts of the country, e.g. rural areas shifting Republican while suburban areas shift Democrat in the USA. While such realigning trends may largely be swamped by election specific tides, in some elections realigning effects will be dominant in at least some parts of the nation (or state).

Note that in addition to the conditions (A1) – (A4), all our basic models (not containing any term involving reversion to the mean) also satisfy:

$$\operatorname{sgn}(s_i) = \operatorname{sgn}(s) \qquad \text{for all } i. \tag{A5}$$

In other words, the district-level swings all have the same sign as the overall swing. The scenario described above shows that this may not always be satisfied, and this is borne out on real data. Electoral tides do not always move in the same direction across all districts – a fact which is concealed by the degree to which prediction errors cancel out. Relaxing the constraint (A5) will therefore probably be necessary.

In the electoral context there are a number of additional substantive factors that in future might usefully be examined, including

- a simple realigning reversal tide effect in which areas previously providing strong support to one party begin to shift in the direction of the other party;
- a polarizing effect, such that areas previously strong for one party generally become stronger still, with the potential, as noted above, for realigning tides to be moving in opposite directions in different parts of a polity;
- an intimidation effect that makes the most successful incumbents less likely to face strong challengers.

The first of these could be represented via a reversion to the mean effect, while the second two involve movements in the other direction. Adding a linear correction, as we discussed in Section 2.2.4, maintains axiomatic properties (provided the slope is of the correct sign) and gives a better fit to data. The question of how to determine the slope of the linear correction in a principled way, in order to capture the interplay of the above three factors, we leave for a companion paper now in preparation.

Conclusion

Given the axiomatic superiority of the piecewise model, its grounding in a substantive theory of electoral change, and the simplicity of the piecewise formula, we recommend its adoption wherever uniform or proportional swing have previously been used. At the very least, anyone using uniform swing should understand its sensitivity to choices made about how to treat data violations of the bounds condition. We agree with Taagepera that it is simply is a bad idea to fit data with a model that is known to generate out of bounds estimates, even if the fit of the more plausible model is only marginally better by various statistical measures. Moreover, even if we shift from a deterministic form of uniform swing to a probabilistic form the problem of out of bounds results does not go away. We make one last observation. This essay is intended to be a theoretical contribution that lays out conditions that any plausible theoretic model of inter-election change should satisfy, in order to move move beyond simple curve-fitting. The empirical analyses we provide are intended to be illustrative. We are preparing a companion paper that argues that the alleged empirical good fit of uniform swing (and other models) depends very much on the measures of fit we use. In particular, the implied assumption (A5), that in each constituency swing is always in the same direction as it is polity wide, is not satisfied in practice, and we need to understand the conditions under which fundamental failure of the uniform swing model can be expected to occur in real data.

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Notes

1. We do have to be careful with nomenclature, since "swing" is also used in the electoral systems and party literatures to refer to "responsiveness", namely the percentage change in seat share for each one percentage point change in vote share (see e.g., Tufte, 1973). That usage of "swing" has been extensively studied both theoretically and empirically. Here we will use the term "swing" only to refer to the magnitude of inter-election shifts in votes.
2. Jonathan Katz (personal communication): "I was not able to find any actual formal definition of proportional swing in the literature ...I wrote the code this way because it works in the sense of making the means correct for both positive and negative swings, which is what is needed."
3. Note that in the literature there are other ways to impute vote shares in uncontested elections, such as looking at presidential or gubernatorial races. Not only would this take us outside the scope of the current paper, it is clear that such a change in methodology would not change the results enough to change our basic conclusion about the relative performance of the three models.

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