

Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior ACOW Virtual Meeting (talk being recorded)

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(joint work with Stephen Melczer)
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Example

- How many n -step lattice walks are there, if walks start from the origin, are confined to the first quadrant, and take steps in $\{S, NE, NW\}$? Call this a_n .
- Now reverse the steps to get $\{N, SE, SW\}$; call the analogous quantity b_n .
- Conjectured by Bostan & Kauers (2009):

$$a_n \sim 3^n \sqrt{\frac{3}{4\pi n}}$$

$$b_n \sim (2\sqrt{2})^n \frac{\theta(n)}{\pi n^2}$$

$$\theta(n) = \begin{cases} 24\sqrt{2} & \text{if } n \text{ is even} \\ 32 & \text{if } n \text{ is odd.} \end{cases}$$

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Overview — walks

- Consider nearest-neighbour walks in \mathbb{Z}^2 , defined by a set $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{\mathbf{0}\}$ of **short steps**.
- We can consider restrictions, e.g. halfspace, nonnegative quadrant, return to x or y -axis, return to the origin.
- We keep track of the endpoint, and also the length. This gives a trivariate sequence $a_{r,s,n}$ with **generating function** (GF)

$$C(x, y, t) := \sum_{r,s,n} a_{r,s,n} x^r y^s t^n.$$

- Summing over r, s gives a univariate series $C(1, 1, t) := f(t) = \sum_n f_n t^n$.
- We seek in particular the asymptotics of f_n .

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Interlude - a hierarchy of generating functions

- Rational functions — constant coefficient linear recurrence for coefficients. Example: Fibonacci numbers.
- Algebraic functions — diagonals of rational functions. Example: Catalan numbers.
- D-finite functions (satisfy linear ODE with polynomial coefficients) — polynomial coefficient linear recurrence for coefficients. Example: $\binom{3n}{n}$, Bessel functions.
- Differentially algebraic functions (nonlinear ODE). Example: Bell numbers (egf).
- Worse! Differentially transcendental. Example: $\Gamma(z)$, Bell numbers (ogf).

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A hierarchy of generating functions from lattice walks

- Unrestricted walks — rational functions — have been understood “forever”.
- Walks confined to a halfspace — algebraic functions — understood since Bousquet-Mélou & Petkovšek (2000), using the *kernel method*.
- 23 classes of walks confined to a quadrant — D-finite functions — reasonably well understood.
- 56 quadrant classes, steps that are not small — non D-finite functions — poorly understood.

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Previous work on walks in the quadrant, I

- Bousquet-Mélou & Mishna (2010): there are 79 inequivalent nontrivial cases.
- They introduced a symmetry group $G(\mathcal{S})$ and showed that it is finite in exactly 23 cases.
- They used finiteness to show for 22 cases that $C(x, y, t)$ is D-finite. For 19 of these, used the *orbit sum method* and for 3 more, the *half orbit sum method*.
- Bostan & Kauers (2010) explicitly showed that for the 23rd case (*Gessel walks*), $f(t)$ is algebraic (and hence D-finite).
- In the other 56 cases, $f(t)$ is indeed apparently not D-finite. So there are 23 nice inequivalent cases to discuss now.

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Standing assumptions

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \dots, z_d)$, $\mathbf{r} = (r_1, \dots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \dots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \rightarrow \mathbb{C}$ for some fixed d . Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The **generating function** (GF) is the formal power series

$$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}.$$

- Assume $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ where G, H are polynomials. The **singular variety** $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$ consists of **poles**.
- To avoid discussing complicated topology, assume all coefficients of F are nonnegative.

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Outline of ACSV project results

- Given a direction $\bar{\mathbf{r}}$, to compute asymptotics of $a_{\mathbf{r}}$ in that direction we first restrict to a variety $\text{crit}(\bar{\mathbf{r}})$ of **critical points**.
- A subset $\text{contrib}(\bar{\mathbf{r}}) \subseteq \text{crit}(\bar{\mathbf{r}})$ contributes to asymptotics.
- For $\mathbf{p} \in \text{contrib}(\bar{\mathbf{r}})$, there is a full asymptotic series $\mathcal{A}(\mathbf{p})$ depending on the type of singularity at \mathbf{p} . Each term is computable from finitely many derivatives of G and H at \mathbf{p} .
- This yields an asymptotic expansion

$$a_{\mathbf{r}} \sim \sum_{\mathbf{p} \in \text{contrib}(\bar{\mathbf{r}})} \mathbf{p}^{-\mathbf{r}} \mathcal{A}(\mathbf{p})$$

that is uniform on compact subsets of directions, provided the geometry at \mathbf{p} does not change.

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Smooth formulae for general d

- \mathbf{z}_* turns out to be a critical point for $\bar{\mathbf{r}}$ iff the outward normal to $\log \mathcal{V}$ is parallel to \mathbf{r} . In other words, for some $\lambda \in \mathbb{C}$, \mathbf{z}_* solves

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \dots, z_d H_d) = \lambda \mathbf{r}, H(\mathbf{z}) = \mathbf{0}.$$

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$$a_{\mathbf{r}} \sim \mathbf{z}_*(\bar{\mathbf{r}})^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2} \kappa(\mathbf{z}_*)} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}}$$

where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.
- The formula comes from analysis of certain highly oscillatory *Fourier-Laplace integrals*.

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where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.
- The formula comes from analysis of certain highly oscillatory *Fourier-Laplace integrals*.

Hörmander's explicit formula

The asymptotic contribution of an isolated nondegenerate stationary point is

$$\left(\det \left(\frac{\lambda f''(\mathbf{0})}{2\pi} \right) \right)^{-1/2} \sum_{k \geq 0} \lambda^{-k} L_k(A, f)$$

where L_k is a differential operator of order $2k$ evaluated at $\mathbf{0}$. Specifically,

$$\underline{f}(t) = f(t) - (1/2)t f''(0) t^T$$

$$\mathcal{D} = \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b} (-i\partial_a)(-i\partial_b)$$

$$L_k(A, f) = \sum_{l \leq 2k} \frac{\mathcal{D}^{l+k}(A \underline{f}^l)(0)}{(-1)^k 2^{l+k} l! (l+k)!}$$

Outline of approach

- Can in fact get results for weighted walks in general dimension.
- Express f_n as diagonal coefficients of $d + 1$ -variable rational GF F , using the kernel method, orbit sum method, and series manipulations.
- Use mvGF theory of Pemantle and Wilson to extract asymptotics.
- Difficulty 1: singular set of F causes problems and F may have nonpositive coefficients.
- Difficulty 2: numerator often vanishes at points contributing to asymptotics, making general formulae hard to derive.
- Solution 1: ask Mireille Bousquet-Mélou!
- Solution 2: work hard.

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General dimension

- Melczer-Mishna analysed the case where \mathcal{S} is symmetric over all d axes.
- We analyse the case with $d - 1$ axes of symmetry (with weights having the same symmetry).
- Examples show that with fewer than $d - 1$ symmetries, the GF is not D-finite, so such an approach must fail.
- We write $S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} w_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = z_d B + Q + \bar{z}_d A$ where $\bar{z} = z^{-1}$ and A, B, Q are independent of z_d .
- The **drift** is the difference $B(\mathbf{1}) - A(\mathbf{1})$ between the weight of positive and negative steps in the asymmetric direction.

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Deriving generating function: kernel method

- Introduced by Knuth and developed by Bousquet-Mélou and others into a powerful tool.
- Recursion gives

$$(1 - tS(\mathbf{z}))z_1 \cdots z_d F(\mathbf{z}, t) = z_1 \cdots z_d + \sum_{k=1}^d L_k(\mathbf{z}_{\hat{k}}, t)$$

where $L_k(\mathbf{z}_{\hat{k}}, t) \in \mathbb{Q}[\mathbf{z}_{\hat{k}}][[t]]$.

- There is a symmetry group of S generated by maps $z_k \mapsto 1/z_k$ and $z_d \mapsto \bar{z}_d \frac{A(\mathbf{z}_{\hat{d}})}{B(\mathbf{z}_{\hat{d}})}$.
- An alternating sum over the group almost fixes the left side and kills the L_k terms on the right, allowing us to solve for the power series F by taking the terms with no negative powers.
- We use a simple change of variable to convert the positive part of a Laurent series to the diagonal of a series.

Theorem

$$F(\mathbf{1}, t) = \Delta \left(\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} \right),$$

where

$$G(\mathbf{z}, t) = (1 + z_1) \cdots (1 + z_{d-1}) (1 - tz_1 \cdots z_d (Q + 2z_d A))$$

$$H(\mathbf{z}, t) = (1 - z_d) \left(1 - tz_1 \cdots z_d \bar{S}(\mathbf{z}) \right) \left(1 - tz_1 \cdots z_d (Q + z_d A) \right),$$

and

$$\bar{S}(\mathbf{z}) = S(\mathbf{z}_{\hat{d}}, \bar{z}_d) = \bar{z}_d B(\mathbf{z}_{\hat{d}}) + Q(\mathbf{z}_{\hat{d}}) + z_d A(\mathbf{z}_{\hat{d}}).$$

Theorem (Positive Drift Asymptotics)

Let

$$b_k = \sum_{\mathbf{i} \in \mathcal{S}, i_k=1} w_{\mathbf{i}} = \sum_{\mathbf{i} \in \mathcal{S}, i_k=-1} w_{\mathbf{i}}.$$

for $1 \leq k < d$. Then

$$f_n \sim S(\mathbf{1})^n \cdot n^{\frac{-(d-1)}{2}} \cdot \left[\left(1 - \frac{A(\mathbf{1})}{B(\mathbf{1})} \right) \left(\frac{S(\mathbf{1})}{\pi} \right)^{\frac{d-1}{2}} \frac{1}{\sqrt{b_1 \cdots b_{d-1}}} \right].$$

Theorem (Negative Drift Asymptotics)

Let $\rho = \sqrt{\frac{A(\mathbf{1})}{B(\mathbf{1})}}$, let $b_k(\mathbf{z}_{\hat{k}}) := [z_k]S(\mathbf{z}) = [z_k^{-1}]S(\mathbf{z})$ and let

$$C_\rho := \frac{S(\mathbf{1}, \rho) \rho}{2 \pi^{d/2} A(\mathbf{1}) (1 - 1/\rho)^2} \cdot \sqrt{\frac{S(\mathbf{1}, \rho)^d}{\rho b_1(\mathbf{1}, \rho) \cdots b_{d-1}(\mathbf{1}, \rho) \cdot B(\mathbf{1})}}.$$

- If $Q \neq 0$ then

$$f_n \sim S(\mathbf{1}, \rho)^n \cdot n^{-d/2-1} \cdot C_\rho.$$

- If $Q = 0$ then

$$f_n \sim n^{-d/2-1} \cdot \left[S(\mathbf{1}, \rho)^n \cdot C_\rho + S(\mathbf{1}, -\rho)^n \cdot C_{-\rho} \right].$$

Example

Consider the model defined by $\mathcal{S} = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$, where the south step $(0, -1)$ has weight $a > 0$ and the north step $(0, 1)$ has weight $b > 0$ (when a and b are integers we can think of having multiple copies of each step with different colours). Then

$$A(x) = a \quad Q(x) = \bar{x} + x \quad B(x) = b$$

and

$$f_n \sim \begin{cases} \left(2 + 2\sqrt{ab}\right)^n \cdot n^{-2} \cdot \frac{2a^{1/4}(1+\sqrt{ab})^2}{\pi b^{3/4}(\sqrt{a}-\sqrt{b})^2} & : b < a \\ (2 + 2a)^n \cdot n^{-1} \cdot \frac{2(1+a)}{\sqrt{a}\pi} & : b = a \\ (2 + a + b)^n \cdot n^{-1/2} \cdot \frac{(a+b)\sqrt{2+a+b}}{b\sqrt{\pi}} & : b > a \end{cases}$$

with the different cases corresponding to negative drift, zero drift, and positive drift.
























S	Asymptotics	S	Asymptotics	S	Asymptotics
	$\frac{4}{\pi} \cdot \frac{4^n}{n}$		$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}$		$\frac{A_n}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{2}{\pi} \cdot \frac{4^n}{n}$		$\frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}}$		$\frac{B_n}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$
	$\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$		$\frac{\sqrt{5}}{3\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{C_n}{\pi} \cdot \frac{(2\sqrt{6})^n}{n^2}$
	$\frac{8}{3\pi} \cdot \frac{8^n}{n}$		$\frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2}$
	$\frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}}$		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2}$
	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{\sqrt{7}}{3\sqrt{3\pi}} \cdot \frac{7^n}{\sqrt{n}}$		$\frac{\sqrt{570-114\sqrt{6}}(24\sqrt{6}+59)}{19\pi} \cdot \frac{(2+2\sqrt{6})^n}{n^2}$
	$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$		$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$
	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{2/3}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}}$		









Table: Asymptotics for the 23 D-finite models.

$$A_n = \begin{cases} 24\sqrt{2} & : n \text{ even} \\ 32 & : n \text{ odd} \end{cases}, \quad B_n = \begin{cases} 12\sqrt{3} & : n \text{ even} \\ 18 & : n \text{ odd} \end{cases}, \quad C_n = \begin{cases} 12\sqrt{30} & : n \text{ even} \\ 144/\sqrt{5} & : n \text{ odd} \end{cases}$$

Extensions

- Small modifications yield results for walks constrained to return to an axis or the origin.
- Walks in Weyl chambers can be treated in this way.
- The zero-drift case is tricky; we worked out the generic case (unpublished) but there are many non-generic subcases.
- The cases with algebraic GFs that have unusual exponents of n in asymptotics seem to correspond to rational GFs with degenerate singularities, which are a nice challenge for the ACSV theory.
- Walks with larger steps usually lead to generating functions that are not D-finite, but in some cases such walks may be tractable (?)

Part of table of results for excursions

\mathcal{S}	Return to x -axis	Return to y -axis	Return to origin
	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$
	$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{8}{\pi} \cdot \frac{4^n}{n^3}$
	$\frac{3\sqrt{6}}{2\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{2\sqrt{6}}{\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{3\sqrt{6}}{\pi} \cdot \frac{6^n}{n^3}$
	$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{128}{27\pi} \cdot \frac{8^n}{n^3}$
	$\frac{3\sqrt{3}}{4\sqrt{\pi}} \frac{3^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$	$\epsilon_n \frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3}$
	$\frac{8}{3\sqrt{\pi}} \frac{4^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	$\delta_n \frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^3}$
	$\frac{5\sqrt{10}}{16\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\frac{\sqrt{2}(1+\sqrt{2})^{3/2}}{\pi} \frac{(2+2\sqrt{2})^n}{n^2}$	$\frac{2(1+\sqrt{2})^{3/2}}{\pi} \frac{(2+2\sqrt{2})^n}{n^3}$
	$\frac{5\sqrt{10}}{24\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{30}}{5\pi} \frac{(2\sqrt{6})^n}{n^2}$	$\delta_n \frac{24\sqrt{30}}{25\pi} \frac{(2\sqrt{6})^n}{n^3}$

Reference links

- [Stephen Melczer's doctoral thesis](#) (good introduction to ACSV, the basis for a Springer [undergraduate text](#) to appear 2020/21).
- R. Pemantle & M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013 (graduate-level text, second edition coming 202x).
- [ACSV project website](#)
- [Sage implementations](#) by Alex Raichev and others

Example (2-D case with no symmetry: $\mathcal{S} = \{N, W, SE\}$)

It turns out that

$$F(t) = \Delta \left(\frac{(x^2 - y)(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))} \right).$$

We decompose

$$\begin{aligned} \frac{(x^2 - y)(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))} &= - \frac{(1 - \overline{xy})(x - y^2)(x + 1)}{(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))} \\ &\quad + \frac{(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - xyt(\overline{y} + y\overline{x} + x))}, \end{aligned}$$

Our usual methods now yield

$$f_n = \frac{3^n}{n^{3/2}} \left(\frac{3\sqrt{3}}{2\sqrt{\pi}} + O(n^{-1}) \right).$$

Publication reform

- Pressure is building for complete conversion of the journal system to open access (e.g. Plan S from European research funders)
- Large commercial publishers have incentives not aligned with scholarship or the interests of readers and authors, and provide overall low quality service for very high prices.
- The journal market is dysfunctional (not properly competitive).
- I am associated with several organizations aiming to improve this: MathOA, Free Journal Network, Publishing Reform Forum. If you would like to help or learn more, please contact me.