What is Analytic Combinatorics (good for)?

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Northeast Combinatorics Network Klatch

What are we talking about?

- The monograph *Analytic Combinatorics* (P. Flajolet and R. Sedgewick, 2009) has been very influential.
- The most common usage of the term refers to the application of complex analytic functions to combinatorial enumeration and discrete probability, using generating function techniques.
- This is the discrete analog of solving differential equations via Fourier or Laplace transform.
- Analysis has many branches: real, complex, functional, differential equations, measure theory, ... There are many possible ways to apply it to combinatorics! Maybe we should have used *holomorphic combinatorics*, but it is too late now.

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The univariate case is quite well understood

- Consider the Fibonacci numbers defined by the usual recurrence relation $a_n = a_{n-1} + a_{n-2}, a_0 = 0, a_1 = 1$.
- Using the generating function $F(x) = \sum_{n \geq 0} a_n x^n$, we translate into a defining equation for the GF, namely

$$F(x) = x + x(F(x) - a_0) + x^2 F(x)$$

and a solution

$$F(x) = \frac{x}{1 - x - x^2}.$$

- Partial fractions and a basic lookup table now give

$$a_n = \frac{1}{\sqrt{5}} \left(\theta^n - (-\theta)^{-n}\right) \sim \frac{1}{\sqrt{5}} \theta^n.$$

- Alternatively we could use residue theory near the dominant pole at $z = 1/\theta$. This generalizes better to higher dimensions.

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Key principles of analytic combinatorics

- Derive a tractable expression (somehow) for the generating function.
- Express the coefficient as a complex integral using the Cauchy Integral Formula:

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(z)\, d\mathbf{z}.$$

- The location of dominant points of the singular variety $\mathcal{V}$ of the GF determines exponential growth rate of coefficients.
- The type of singularity determines the polynomial correction factors.
- Using these has allowed for a huge number of applications, especially for rational, algebraic and D-finite GFs. See Flajolet-Sedgewick for the univariate case.

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Multivariate case

- Deriving multivariate GFs is often not much harder than in the univariate case.
- However analysing them *is* much harder, even for rational functions:
  - Algebra: partial fraction decomposition does not apply in general.
  - Geometry: the singular variety $\mathcal{V}$ does not consist of isolated points, and may self-intersect.
  - Topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated.
- Analysis: the (Leray) residue formula is much harder to use.
- Computation: harder owing to the curse of dimensionality.

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Example (Delannoy walks)

- We count walks on the lattice $\mathbb{Z}^2$ starting at $(0,0)$ and ending at (r,s), with each step chosen from $\{\uparrow, \rightarrow, \nearrow\}$.
- The recurrence is

$$a_{rs} = \begin{cases} a_{r-1,s-1} + a_{r-1,s} + a_{r,s-1} & \text{if } r > 1, s > 1 \\ 1 & (r,s) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

and GF is $(1 - x - y - xy)^{-1}$.
- How to compute asymptotics for $a_{r,s}$ for large r, s?

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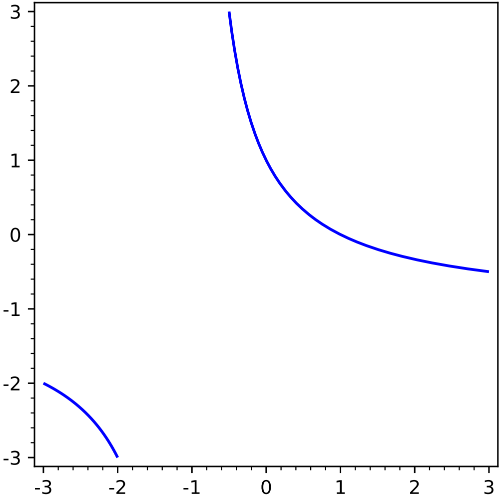
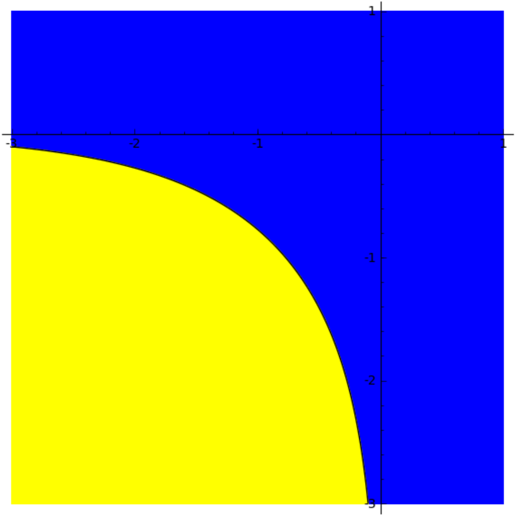
Example (Delannoy walks: singular variety)

The complex curve given by $1 - x - y - xy = 0$ (real points shown).



Example (Delannoy walks: singular variety in log coordinates)



ACSV project: outline of generic combinatorial case

- We want asymptotics of $a_{\mathbf{r}}$, having GF $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.
- In direction $\overline{\mathbf{r}}$ these are determined by the geometry of $\mathcal{V}$ near an explicit variety, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- We may restrict to a dominant contributing point $\mathbf{z}_*(\overline{\mathbf{r}})$ lying in the positive orthant.
- There is an asymptotic series $\mathcal{A}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, depending on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields

$$a_{\mathbf{r}} \sim \mathcal{A}(\mathbf{z}_*(\mathbf{r}))$$

  where the expansion is uniform on compact cones of directions, provided the geometry does not change.
- The formulae are symbolic and can be used for numerical approximation.

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2008 Twenty combinatorial examples (P.-W., SIAM Rev.)
2008 Higher order asymptotics (Raichev-W., Elec. J. Combin.)
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ACSV project highlights - recent stuff

2019-2021 Books by Mishna and Melczer (see reference slide).

2022 Revised second edition of P.-W. monograph due, with Melczer as coauthor.

2020-2021 AMS Math Research Community for grads/postdocs.

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Sample formulae

- For $U \subseteq \mathbb{C}^d$, define $\log U = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$.
- For smooth contributing points,

$$a_{\mathbf{r}} \sim \mathbf{p}^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{p})}} \frac{G(\mathbf{p})}{|\nabla_{\log} H(\mathbf{p})|}$$

  where $|\mathbf{r}| = \sum_i r_i$, κ is the Gaussian curvature of $\log \mathcal{V}$, and $\nabla_{\log} H(\mathbf{z})$ is the coordinatewise product of $\mathbf{z}$ with $\nabla H(\mathbf{z})$.
- For a complete intersection, there is a cone $\mathrm{K}(\mathbf{p})$, and for directions uniformly in compact subcones of the interior of $\mathrm{K}(\mathbf{p})$,

$$a_{\mathbf{r}} = \mathbf{p}^{-\mathbf{r}} \left[\frac{G(\mathbf{p})}{\det J(\mathbf{p})} + O(e^{-c|\mathbf{r}|}) \right]$$

  where J is the Jacobian formed by the d local factors of H.
- The quantities involved can all be computed as rational expressions in the first and second derivatives of H.

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- For $U \subseteq \mathbb{C}^d$, define $\log U = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$.
- For smooth contributing points,

$$a_{\mathbf{r}} \sim \mathbf{p}^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{p})}} \frac{G(\mathbf{p})}{|\nabla_{\log} H(\mathbf{p})|}$$

  where $|\mathbf{r}| = \sum_i r_i$, κ is the Gaussian curvature of $\log \mathcal{V}$, and $\nabla_{\log} H(\mathbf{z})$ is the coordinatewise product of $\mathbf{z}$ with $\nabla H(\mathbf{z})$.
- For a complete intersection, there is a cone $\mathrm{K}(\mathbf{p})$, and for directions uniformly in compact subcones of the interior of $\mathrm{K}(\mathbf{p})$,

$$a_{\mathbf{r}} = \mathbf{p}^{-\mathbf{r}} \left[\frac{G(\mathbf{p})}{\det J(\mathbf{p})} + O(e^{-c|\mathbf{r}|}) \right]$$

  where J is the Jacobian formed by the d local factors of H.
- The quantities involved can all be computed as rational expressions in the first and second derivatives of H.

Delannoy walk asymptotics

- Uniformly for $r/s, s/r$ away from 0

$$a_{rs} \sim \left[\frac{r}{\Delta - s}\right]^r \left[\frac{s}{\Delta - r}\right]^s \sqrt{\frac{rs}{2\pi\Delta(r+s-\Delta)^2}}.$$

where $\Delta = \sqrt{r^2 + s^2}$.

- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.
- Vastly many problems involving walks, sequences, sums of IID random variables are of similar difficulty level.
- If you know of the "diagonal method" (e.g. in Stanley EC1), note that it fails in almost all such cases.

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Delannoy walk numerics

Extracting a given diagonal is now easy: for example $a_{13n,19n} \sim AC^n n^{-1/2}$ where

$$C = \frac{59921406409232595462295329086621793468\,7}{(\sqrt{530}-13)^{19}(\sqrt{530}-19)^{13}}$$

$$\approx 7.98556079229048 \times 10^{11}$$

$$A \approx 0.145544855274974.$$

- The relative error even for $n = 1$ is under 1% and error decreases as $1/n$.
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Key uses of these methods

- Precise asymptotics
- Accurate numerical approximations
- Disproving positivity conjectures
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Nearest-neighbor walks confined to positive quadrant - Melczer & Wilson SIAM J. Disc. Math. 2019)

$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics
	$\frac{4}{\pi} \cdot \frac{4^n}{n}$		$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}$		$\frac{A_n}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{2}{\pi} \cdot \frac{4^n}{n}$		$\frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}}$		$\frac{B_n}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$
	$\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$		$\frac{\sqrt{5}}{3\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{C_n}{\pi} \cdot \frac{(2\sqrt{6})^n}{n^2}$
	$\frac{8}{3\pi} \cdot \frac{8^n}{n}$		$\frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2}$
	$\frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}}$		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2}$
	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{\sqrt{7}}{3\sqrt{3\pi}} \cdot \frac{7^n}{\sqrt{n}}$		$\frac{\sqrt{570-114\sqrt{6}}(24\sqrt{6}+59)}{19\pi} \cdot \frac{(2+2\sqrt{6})^n}{n^2}$
	$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$		$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$
	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{2/3}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}}$		

Table: Asymptotics for the 23 D-finite models.

$$A_n = \begin{cases} 24\sqrt{2} & : n \text{ even} \\ 32 & : n \text{ odd} \end{cases}, \quad B_n = \begin{cases} 12\sqrt{3} & : n \text{ even} \\ 18 & : n \text{ odd} \end{cases}, \quad C_n = \begin{cases} 12\sqrt{30} & : n \text{ even} \\ 144/\sqrt{5} & : n \text{ odd} \end{cases}$$

Some combinatorial GFs with contributing smooth points

- Let a_{rs} be the number of ways to write the highest weight of a type B_r simple Lie algebra as a sum of s positive roots. Then

$$F(x,q) = \sum_{r,s} a_{rs} x^r q^s = \frac{qx - q(1+q)x^2 + q^2x^3}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2}.$$

- The Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ satisfy

$$\sum_{\lambda}\sum_{\mu,\nu}\left(c^{\lambda}_{\mu\nu}\right)^2 q^{|\mu|}t^{|\nu|} = \prod_{i=1}^{\infty}\left(1-q^i-t^i\right)^{-1}.$$

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Example (Narayana numbers)

- These count rooted ordered trees by edges and leaves.
- The bivariate GF $w := F(x, y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(w, x, y) := w^2 - w[1 + x(y-1)] + xy = 0$. Using a known construction (Safonov) we obtain the rational GF

$$G(u, x, y) = \frac{u(1 - 2u - ux(1-y))}{1 - u - xy - ux(1-y)}.$$

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Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way, but the first term in the series is zero, so we need to go further. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

  and we obtain asymptotics starting with s^{-2}. For example

$$a_{2s,s} \sim \frac{16^s}{8\pi s^2}.$$

- This example shows:
  - we can go beyond rational and even meromorphic GFs;
  - we may need to consider noncombinatorial GFs;
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Example (lemniscate)

- Consider $F = 1/H$ where

$$H(x,y) = x^2y^2 - 2xy(x+y) + 5\,(x^2+y^2) + 14\,xy - 20(x+y) + 19.$$

This is combinatorial, and H is an irreducible polynomial.

- All points except $(1,1)$ are smooth, and $(1,1)$ is a transverse strictly minimal double point.
- In the cone $1/2 < r/s < 2$ we have $a_{rs} \sim 6$, outside we use the smooth point formula.
- Note that H factors locally at $(1,1)$ but not globally.

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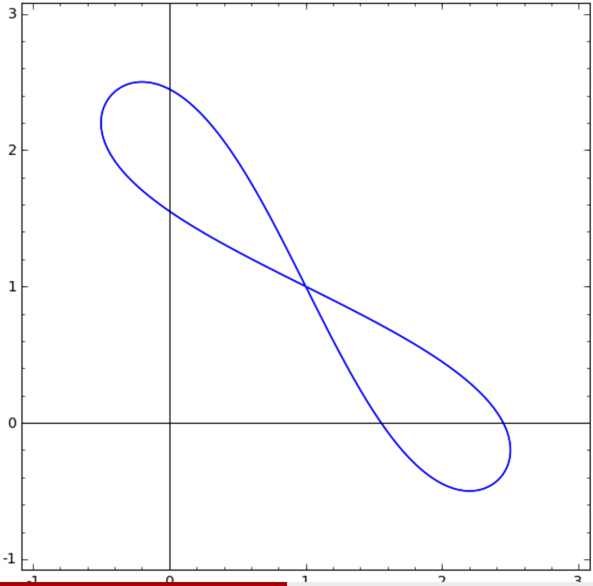
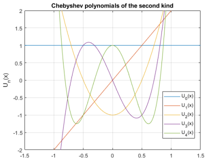
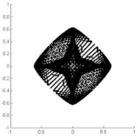
$\mathcal{V}$ for lemniscate

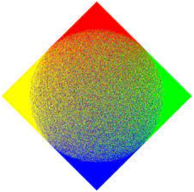


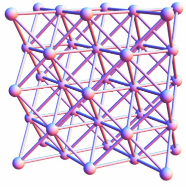
Some harder applications

- Chebyshev polynomials (smooth, two contributing critical points, periodicity)
- quantum random walks (smooth, non-isolated critical points)
- cube groves, frozen regions for tiling models in statistical mechanics (cone singularities)
- lattice Green's functions (non-transverse multiple points - don't know how to do yet!)

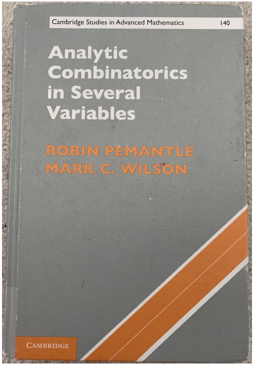


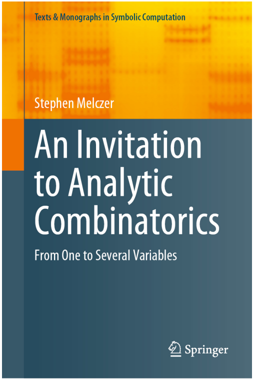


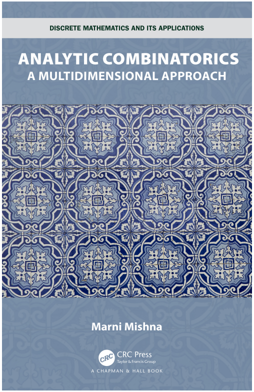




References







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Publication reform

- Pressure is building for complete conversion of the journal system to open access (e.g. Plan S from European research funders)
- Large commercial publishers have incentives not aligned with scholarship or the interests of readers and authors, and provide overall low quality service for very high prices.
- The journal market is dysfunctional (not properly competitive).
- I am associated with several organizations aiming to improve this: MathOA, Free Journal Network, Publishing Reform Forum. If you would like to help or learn more, please contact me.
- Support the journals Algebraic Combinatorics (NOT the zombie J. Alg Comb) and Combinatorial Theory (NOT the zombie JCTA).