Superalgebras and their uses

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Outline

- What is a superalgebra?
- Representation theory
- Lie groups, Lie algebras
- Relations with particle physics
- Lie superalgebras
- My work

Even and odd functions

- Every $f : \mathbb{R} \to \mathbb{R}$ has a unique decomposition $f = f_0 + f_1$ where f_0 is *even* ($f_0(x) = f_0(-x)$) and f_1 is *odd* ($f_1(x) = f_1(-x)$).
- Explicitly,
  $f_0(x) = \frac{1}{2}[f(x) + f(-x)], f_1(x) = \frac{1}{2}[f(x) - f(-x)]$.
- Example:
  $e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \cosh(x) + \sinh(x)$.
- Multiplication: odd · odd=even, odd · even=odd, even · even=even.
- Useful for simplifying differentiation and integration formulae, since derivative of even is odd, etc.

Superalgebra

- Extract essential structure of previous example.
- An *algebra* is a vector space with a (bilinear) multiplication. Examples: all $n \times n$ matrices, $\mathbb{R}^3$ under cross product.
- A *superalgebra* is an algebra with vector space decomposition $A = A_0 \oplus A_1$, satisfying the multiplication rules for even and odd elements in previous example.
- Here elements of A_0 are called even, those of A_1 odd.

Another view of superalgebras

- Given $a = a_0 + a_1 \in A$, can recapture a_0, a_1 as before. Define $\sigma(a) = \sigma(a_0 + a_1) = a_0 - a_1$. Then $a_0 = \frac{1}{2}(a + \sigma(a)), a_1 = \frac{1}{2}(a - \sigma(a))$.
- In fact σ is an automorphism of A and $\sigma^2 = 1$.
- Can give alternative definition of superalgebra: an algebra with an automorphism σ of order 2. As a linear map σ has a $+1$ eigenspace A_0 and a -1 eigenspace A_1 and this gives the original definition.

An example

$gl(m,n)$ is the algebra of block matrices of the form

$$\begin{pmatrix} m \times m & m \times n \\ n \times m & n \times n \end{pmatrix}.$$

The even part is the matrices of the form

$$\begin{pmatrix} m \times m & 0 \\ 0 & n \times n \end{pmatrix}$$

and the odd of matrices of the form

$$\begin{pmatrix} 0 & m \times n \\ n \times m & 0 \end{pmatrix}.$$

Key example: Grassmann algebra

- $\Lambda(n) = \mathbb{C}\langle x_1, \ldots, x_n \mid x_i x_j = -x_j x_i \rangle$.
- The algebra of "polynomials" in n "anticommuting" variables.
- Example for $n = 3$: $zxy = -xzy = +xyz$, $x^2 = y^2 = z^2 = 0$. Basis: $\{1, x, y, z, xy, xz, yz, xyz\}$.
- Monomials of even length commute with everything, those of odd length anticommute.
- $\Lambda_0 =$ span of monomials of even length, $\Lambda_1 =$ span of monomials of odd length.

- The Grassmann algebra is also called the exterior algebra on the vector space with basis $x_1, \ldots, x_n$.
- It yields the slickest way to define determinants.
- It is of fundamental importance in
  - differential geometry (differential forms)
  - invariant theory
  - study of identical relations in algebra (e.g. Burnside problem).
- It is the most important superalgebra, and is useful in most antisymmetric situations.

Representation theory I

- In classical times, all algebraic objects were thought of concretely (e.g. groups of permutations or invertible matrices). The concept of abstract algebraic structures had not arisen.
- In this century, much abstraction occurred (axiomatic approach to groups, rings, fields, etc). This enabled much more progress to be made (e.g. classification of finite simple groups) by concentrating on the essentials.
- However, in applications we still usually want concrete information about, say, a group of matrices.

Representation theory II

- Thus there is a “division of labo(u)r”: study abstract objects / study all ways a given such object can arise concretely. There is much interplay between the two.
- We use *linear* representations so we can use linear algebra machinery (e.g. eigenvalues).
- Choose a vector space V of dimension n say. To each element in the algebraic object we associate a linear transformation of V, in a consistent way (a homomorphism).

- Some information may be lost this way but considering *all* representations together gives the whole picture, usually.
- Analogy: photographs or cross-sections of a 3-dimensional object. No one of them gives the full picture, but taken together they do.
- Particularly important are *irreducible* representations. These are (usually) the “building blocks”.
- Irreducible representations of discrete groups are heavily used by crystallographers and chemists, among others.

Lie groups

- These are *continuous* groups often arising as symmetry groups of physical systems.
- They are ubiquitous in mathematics, in areas from differential equations to number theory.
- Example: $SU(3)$, group of all 3×3 Hermitian (i.e. $\overline{X}^t = X$) matrices of determinant 1.
- The irreducible representations of $SU(3)$ have been used by physicists (the "eightfold way") to study elementary particles.

Lie algebras

- Most information about representations of a Lie group can be obtained by looking at its linearization, the *Lie algebra*.
- In fact representations of the group are in 1-1 correspondence with the representations of the Lie algebra.
- Lie algebras can be studied combinatorially, even using computer programs. There is an extensive, rather well-understood theory.
- “Every issue” of J. Nuclear Physics B contains Lie algebra calculations.

Particle physics

- Dirac unified quantum mechanics and special relativity. This required the introduction of a new symmetry, between particles and *antiparticles*.
- This was a source of consternation (only electron, proton and neutron were known).
- Experiment then found positron, and then huge numbers of new "elementary" particles.
- Gell-Mann et al. introduced quarks as a way of explaining these particles (the eightfold way).

Supersymmetry

- Attempts to improve the “standard model” of particle physics have led to “superstring” theory. It again implies a new symmetry of nature, known as *supersymmetry*.
- This symmetry interchanges *bosons* (force-carrying particles like photons) with *fermions* (matter particles like electrons and quarks). It requires each known particle to have a super-partner (none of these have yet been observed).
- The mathematical formulation uses *Lie superalgebras* in an essential way (this is the origin of the name “superalgebra”).

Lie algebras and superalgebras

- A *Lie algebra* is an algebra whose multiplication $[\,,\,]$ satisfies $[x, y] = -[y, x]$ and the Jacobi identity $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$.
- A *Lie superalgebra* is a superalgebra whose multiplication satisfies

$$[x, y] = \begin{cases} -[y, x] & \text{if } x \text{ or } y \text{ is even,} \\ +[y, x] & \text{if } x \text{ and } y \text{ are odd.} \end{cases}$$

  and a "super" version of the Jacobi identity.

There are many evident similarities between the two.

Differences and difficulties

- Lie superalgebras exhibit several structural features which are more complicated than the Lie algebra situation. The main ones are:
    1. A given representation might not break up into a direct sum of irreducible ones.
    2. Putting all the irreducibles together might not give a complete picture of the algebra.
- These problems can be studied in the common framework of ring theory.
- The key technical tool is the *enveloping algebra* of the Lie (super)algebra.

My work in the area

- The *simple* Lie superalgebras have been classified by V. Kac.
- A criterion for when difficulty (2) does not arise was given by A. Bell in 1990.
- I applied Bell's criterion to several infinite families of simple Lie superalgebras and showed it did not work for others.
- This was a multi-paper, multi-author project which required many different techniques including some computer experimentation. Interesting questions remain.

Example

$W(n)$ is the span of all "Grassmann vector fields" $P(x_1, x_2, \ldots, x_n)\partial/\partial x_i \equiv P\partial_i$. Multiplication is $[P\partial_i, Q\partial_j] = P\partial_i(Q)\partial_j \pm Q\partial_j(P)\partial_i$. Dimension is $n2^n$ and this is a simple Lie superalgebra. It is the analogue of a simple infinite-dimensional Lie algebra of Cartan, which uses commuting, not anticommuting variables.

Interesting ring-theoretic questions

- For $L = W(2n)$, I have shown that $J(U(L)) = 0$ but it is known that the intersection of annihilators of finite-dimensional irreducibles is nonzero. This doesn't happen for Lie algebras. What is a nice subset of irreducibles with annihilator zero?
- Primitive ideals of $U(L)$ is the next step. Huge theory for Lie algebras.
- Does $U(L)$ always have a unique minimal prime?
- Converse of Bell's criterion ($W(2n+1)$ in particular)?