Diagonal asymptotics for products of combinatorial classes
Or: the diagonal method is still not very good

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- No Maple package, but there is now a reasonable implementation in Sage (available at Alex's website). Needs some algorithmic speedups. Any volunteers?
- In 2012, I saw that the word has not yet spread far enough. Multivariate methods are more general, conceptually simpler, and, I claim, computationally superior.

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- The answer can be given explicitly in this case: $\sum_k \binom{n-1}{k}^2 = \binom{2n-2}{n-1}$. Thus

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- Suppose we replace "two" by d, $\mathbb{N}$ by other combinatorial classes, allow different n for different compositions,...?

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- Banderier & Hitczenko 2012: generalize from 2 to d compositions, different restriction S for each one. Some explicit formulae and asymptotics.

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- Use the symbolic method. Let $F(\mathbf{x}, \mathbf{y}) = \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \mathbf{y}^{\mathbf{k}}$ be the $2d$-variate generating function, where $\mathbf{x}$ marks size and $\mathbf{y}$ marks number of components. Here $F(\mathbf{x}, \mathbf{y})$ factors as $\prod_{i=1}^{d} F_i(x_i, y_i)$.

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- The number of d-tuples of objects with the same number of components is $[\mathbf{x}^{\mathbf{n}}] \operatorname{diag}_{\mathbf{y}} F(\mathbf{x}, \mathbf{1})$. In particular for the simplest case where all $n_i = n$,

$$[\mathbf{x}^{n\mathbf{1}}] \operatorname{diag}_{\mathbf{y}} F(\mathbf{x}, 1) = \sum_{k \geq 0} (a_{nk})^d =: b_n.$$

Aside: exact solutions

- When $d = 2$, we have a good chance of finding an exact solution. For Dyck walks

$$\sum_{\substack{0 \le k \le n \\ 2 | (n-k)}} \left[\frac{k+1}{n+1} \binom{n+1}{\frac{n-k}{2}} \right]^2 = \frac{1}{n+1} \binom{2n}{n}.$$

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- When $d \ge 3$, exact solutions are rare. For example, $b_n = \sum_k \binom{n}{k}^3$ is known not to have an algebraic generating function.

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- It seems that the work needed is enormous even for rather modest-looking problems. For example, the defining linear differential equation for $\sum_k \binom{n-k}{k}^5$ has order 6 with polynomial coefficients of degree 38. Banderier and Hitczenko report: "Current state of the art algorithms will take more than one day for $d = 6$, and gigabytes of memory .... "

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- How to do it for general d? Also, the diagonal method does not yield asymptotics that are uniform in the slope of the diagonal; performance away from the main diagonal is bad.

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- In general, such methods say nothing about higher order terms, or when there is not a continuous limit. Still, this approach is a useful complement to the above methods.

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  - We want numerical approximations for smaller values of $\mathbf{n}$.
- This topic was the subject of two papers with Alex Raichev. For example, our 2nd order approximation for $\sum_{k=0}^{n} \binom{n}{k}^5$, even for $n = 8$, has relative error only 0.5%, but 10% for 1st order.

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- For more, see the book (next talk!).

General asymptotic formula (supercritical Riordan case)

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- Suppose $F_i(x, y) = \phi(x)/(1 - yv(x))$ and ϕ has radius of convergence large enough. Let $c > 0$ solve $v(c) = 1$. Then

$$b_{n\mathbf{1}} \sim c^{-dn} n^{-d/2} \sum_l c_l n^{-l}$$ where c_l is explicitly computable.

In particular

$$c_0 = \frac{\phi(c)^d}{\sqrt{d}\mu_v(c)\left[2\pi \frac{\sigma_v^2(c)}{\mu_v(c)}\right]^{\frac{d-1}{2}}}.$$

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- $$\sum_{k\geq 0} \binom{6n}{k} \binom{3n}{k} \binom{2n}{k} \sim \left(\frac{524288}{729} \right)^{n} \left[\frac{4\sqrt{11}}{33\pi n} - \frac{5446}{395307} \frac{\sqrt{11}}{\pi n^{2}} \right].$$

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- Once the diagonal GF is found, the asymptotic extraction is quicker, since it is a univariate problem. The multivariate method typically requires solving systems of algebraic equations.
- I suggest a serious theoretical and experimental comparison of the performance of these methods. If done experimentally, we need to implement the methods equally. I know which one I would bet on to win!