Electoral engineering — simulating vote distributions

Mark C. Wilson (joint with G. Pritchard)
www.cs.auckland.ac.nz/~mcw/blog/

Department of Computer Science
University of Auckland
supported by Marsden Fund of NZ

INFORMS International, Hawaii, 2016-06-13

Electoral system design

- There are many important features of parliamentary electoral systems.

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:
  - drawing of electoral district boundaries;

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:
  - drawing of electoral district boundaries;
  - rules for nomination of candidates;

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:
  - drawing of electoral district boundaries;
  - rules for nomination of candidates;
  - voter registration procedures;

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:
  - drawing of electoral district boundaries;
  - rules for nomination of candidates;
  - voter registration procedures;
  - design of ballots;

Electoral system design

- There are many important features of parliamentary electoral systems.
- We consider here only the electoral formula which translates votes into seats in a legislature.
- Other features not considered here include:
  - drawing of electoral district boundaries;
  - rules for nomination of candidates;
  - voter registration procedures;
  - design of ballots;
  - rules for counting ballots.

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);
  - algorithms (vote aggregation rules);

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);
  - algorithms (vote aggregation rules);
  - performance measures (indices of proportionality and party system fragmentation).

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);
  - algorithms (vote aggregation rules);
  - performance measures (indices of proportionality and party system fragmentation).
- We aim to distinguish between reasonable and clearly Pareto-suboptimal parameter settings.

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);
  - algorithms (vote aggregation rules);
  - performance measures (indices of proportionality and party system fragmentation).
- We aim to distinguish between reasonable and clearly Pareto-suboptimal parameter settings.
- We (Fowlie & Wilson 2012) have done this on a much smaller scale in the context of a review of the NZ voting system.

Overview of larger research project

- We aim for comprehensive simulation analysis of electoral systems, focusing on:
  - inputs (vote/preference distributions);
  - algorithms (vote aggregation rules);
  - performance measures (indices of proportionality and party system fragmentation).
- We aim to distinguish between reasonable and clearly Pareto-suboptimal parameter settings.
- We (Fowlie & Wilson 2012) have done this on a much smaller scale in the context of a review of the NZ voting system.
- We focus only on families of systems based on plurality ballots (vote for a single candidate/party).

Key parameters

- The district magnitude is the number of representatives elected from a district. It ranges from 1 (very common in USA) to the entire parliament size (e.g. proportional representation in Netherlands).

Key parameters

- The district magnitude is the number of representatives elected from a district. It ranges from 1 (very common in USA) to the entire parliament size (e.g. proportional representation in Netherlands).
- The rounding method is the way integrality constraints on numbers of seats by parties are dealt with. The main ones are named for Webster/St Lagüe and Jefferson/d'Hondt. The choice does make a difference, but it is less important than district magnitude or thresholds.

Key parameters

- The district magnitude is the number of representatives elected from a district. It ranges from 1 (very common in USA) to the entire parliament size (e.g. proportional representation in Netherlands).
- The rounding method is the way integrality constraints on numbers of seats by parties are dealt with. The main ones are named for Webster/St Lagüe and Jefferson/d'Hondt. The choice does make a difference, but it is less important than district magnitude or thresholds.
- The threshold below which votes for a party are treated as zero. This is as high as 10% in Turkey and as low as 0% in Netherlands, but is commonly around 3–5%.

Algorithms — plurality based families

- simple family: n/d districts of magnitude d;

Algorithms — plurality based families

- simple family: n/d districts of magnitude d;
- proportional representation (PR): 1 district with magnitude n;

Algorithms — plurality based families

- simple family: n/d districts of magnitude d;
- proportional representation (PR): 1 district with magnitude n;
- First Past the Post (FPP): n districts of magnitude 1;

Algorithms — plurality based families

- simple family: n/d districts of magnitude d;
- proportional representation (PR): 1 district with magnitude n;
- First Past the Post (FPP): n districts of magnitude 1;
- MM: fraction α of seats allocated by FPP, the rest by PR using the same ballot;

Algorithms — plurality based families

- simple family: n/d districts of magnitude d;
- proportional representation (PR): 1 district with magnitude n;
- First Past the Post (FPP): n districts of magnitude 1;
- MM: fraction α of seats allocated by FPP, the rest by PR using the same ballot;
- Multi-level systems: voters give a single district vote. At each level, districts are aggregated along with corresponding votes, and some seats allocated. With 1 level this is FPP and with 2 levels it is MM.

Measures

- A standard measure of stability of the political system is the effective number of parties (Laakso & Taagepera, 1979). If p is the probability distribution of seats over parties in parliament, then

$$L(p) := \frac{1}{\sum_i p_i^2}.$$

Measures

- A standard measure of stability of the political system is the effective number of parties (Laakso & Taagepera, 1979). If p is the probability distribution of seats over parties in parliament, then
$$L(p) := \frac{1}{\sum_i p_i^2}.$$
- Taagepera (1987) shows that expected cabinet duration is well fitted by $42/L^2$.

Measures

- A standard measure of stability of the political system is the effective number of parties (Laakso & Taagepera, 1979). If p is the probability distribution of seats over parties in parliament, then
$$L(p) := \frac{1}{\sum_i p_i^2}.$$
- Taagepera (1987) shows that expected cabinet duration is well fitted by $42/L^2$.
- The standard measure of proportionality for party-based systems is the Gallagher least squares index. If v_i denotes the vote share, then
$$G(p, v) = \left(\sum_i (v_i - p_i)^2 \right)^{1/2}.$$

The single-district problem

- There are many standard distributions used for studying single-winner voting systems.

The single-district problem

- There are many standard distributions used for studying single-winner voting systems.
- The most commonly used method in the literature is simply to make iid draws from the uniform distribution on possible votes (Impartial Culture).

The single-district problem

- There are many standard distributions used for studying single-winner voting systems.
- The most commonly used method in the literature is simply to make iid draws from the uniform distribution on possible votes (Impartial Culture).
- This has been severely criticised for lack of realism (Regenwetter *et al.* 2008).

The single-district problem

- There are many standard distributions used for studying single-winner voting systems.
- The most commonly used method in the literature is simply to make iid draws from the uniform distribution on possible votes (Impartial Culture).
- This has been severely criticised for lack of realism (Regenwetter *et al.* 2008).
- For plurality ballots, we are sampling from ordered m-compositions of N where m is the number of candidates and N the number of voters in the district.

Another approach — single district

- In the single-district case, one approach has been to use a Polya urn model and Polya-Eggenberger distribution.

Another approach — single district

- In the single-district case, one approach has been to use a Polya urn model and Polya-Eggenberger distribution.
- Starting with a single ball for each party in an urn, we draw a ball uniformly at random and replace it in the urn together with c balls of the same party.

Another approach — single district

- In the single-district case, one approach has been to use a Polya urn model and Polya-Eggenberger distribution.
- Starting with a single ball for each party in an urn, we draw a ball uniformly at random and replace it in the urn together with c balls of the same party.
- This leads to a "rich gets richer" phenomenon, leading to disparity in party sizes.

Another approach — single district

- In the single-district case, one approach has been to use a Polya urn model and Polya-Eggenberger distribution.
- Starting with a single ball for each party in an urn, we draw a ball uniformly at random and replace it in the urn together with c balls of the same party.
- This leads to a "rich gets richer" phenomenon, leading to disparity in party sizes.
- If $c = 0$, we get IC. If $c = 1$ we obtain the Impartial Anonymous Culture: every ordered m-composition of N is equally likely.

Another approach — single district

- In the single-district case, one approach has been to use a Polya urn model and Polya-Eggenberger distribution.
- Starting with a single ball for each party in an urn, we draw a ball uniformly at random and replace it in the urn together with c balls of the same party.
- This leads to a "rich gets richer" phenomenon, leading to disparity in party sizes.
- If $c = 0$, we get IC. If $c = 1$ we obtain the Impartial Anonymous Culture: every ordered m-composition of N is equally likely.
- Larger values of c model more homogeneous societies.

Multiple districts — uniform generation

- For plurality-based systems we can assume that the vote matrix (rows are districts, columns are parties) is uniformly generated from all possible such matrices.

Multiple districts — uniform generation

- For plurality-based systems we can assume that the vote matrix (rows are districts, columns are parties) is uniformly generated from all possible such matrices.
- Of course there are some constraints. In particular the row sums must be approximately equal (districts are usually of roughly the same population).

Multiple districts — uniform generation

- For plurality-based systems we can assume that the vote matrix (rows are districts, columns are parties) is uniformly generated from all possible such matrices.
- Of course there are some constraints. In particular the row sums must be approximately equal (districts are usually of roughly the same population).
- If we don't specify constraints on column sums (national party vote totals), and generate district results independently, the law of large numbers shows that column totals will be very similar. This is highly unrealistic, since party size distribution is usually very far from this.

Multiple districts — uniform generation

- For plurality-based systems we can assume that the vote matrix (rows are districts, columns are parties) is uniformly generated from all possible such matrices.
- Of course there are some constraints. In particular the row sums must be approximately equal (districts are usually of roughly the same population).
- If we don't specify constraints on column sums (national party vote totals), and generate district results independently, the law of large numbers shows that column totals will be very similar. This is highly unrealistic, since party size distribution is usually very far from this.
- Also, real party votes show substantial correlation between districts. In fact, districts are often drawn in order to create clear winners.

Contingency tables

- We could model party sizes (column totals) and then generate uniformly from all constrained matrices.

Contingency tables

- We could model party sizes (column totals) and then generate uniformly from all constrained matrices.
- The above considerations lead to uniform sampling from contingency tables with given margins.

Contingency tables

- We could model party sizes (column totals) and then generate uniformly from all constrained matrices.
- The above considerations lead to uniform sampling from contingency tables with given margins.
- This is a very interesting problem. However there are no standard algorithms known to work well for large numbers of rows (hundreds) and columns (about 10).

Contingency tables

- We could model party sizes (column totals) and then generate uniformly from all constrained matrices.
- The above considerations lead to uniform sampling from contingency tables with given margins.
- This is a very interesting problem. However there are no standard algorithms known to work well for large numbers of rows (hundreds) and columns (about 10).
- A recently introduced recursive algorithm (DeSalvo & Zhao, 2015) may be useful. However we have not pursued this yet.

Multiple districts — urn approach

- We start with initial conditions in the vote matrix (a few voters of each type in each district). Urns are indexed by districts.

Multiple districts — urn approach

- We start with initial conditions in the vote matrix (a few voters of each type in each district). Urns are indexed by districts.
- We randomly choose a district i. With probability p we choose a random ball and replace it in its urn along with c balls of that party.

Multiple districts — urn approach

- We start with initial conditions in the vote matrix (a few voters of each type in each district). Urns are indexed by districts.
- We randomly choose a district i. With probability p we choose a random ball and replace it in its urn along with c balls of that party.
- However with probability $1 - p$ we choose another district j at random to "imitate". We choose instead from that district but add the c balls of that party to district i.

Multiple districts — urn approach

- We start with initial conditions in the vote matrix (a few voters of each type in each district). Urns are indexed by districts.
- We randomly choose a district i. With probability p we choose a random ball and replace it in its urn along with c balls of that party.
- However with probability $1 - p$ we choose another district j at random to "imitate". We choose instead from that district but add the c balls of that party to district i.
- The basic model is of new voters making up their mind under influence from others, who are more likely, but not certain to be in the same district.

Multiple districts — urn approach

- We start with initial conditions in the vote matrix (a few voters of each type in each district). Urns are indexed by districts.
- We randomly choose a district i. With probability p we choose a random ball and replace it in its urn along with c balls of that party.
- However with probability $1-p$ we choose another district j at random to "imitate". We choose instead from that district but add the c balls of that party to district i.
- The basic model is of new voters making up their mind under influence from others, who are more likely, but not certain to be in the same district.
- The initial conditions and the values of p and c are parameters we can attempt to tune to get "realistic" results.

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.
- Taagepera (2007) has a model which yields the following estimates, where M is the mean district magnitude and S the number of legislative seats:

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.
- Taagepera (2007) has a model which yields the following estimates, where M is the mean district magnitude and S the number of legislative seats:
  - The number of vote-getting parties is about $(MS)^{1/4}$.

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.
- Taagepera (2007) has a model which yields the following estimates, where M is the mean district magnitude and S the number of legislative seats:
  - The number of vote-getting parties is about $(MS)^{1/4}$.
  - L is about $(MS)^{1/6}$.

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.
- Taagepera (2007) has a model which yields the following estimates, where M is the mean district magnitude and S the number of legislative seats:
  - The number of vote-getting parties is about $(MS)^{1/4}$.
  - L is about $(MS)^{1/6}$.
  - The maximum vote share of a party is about $(MS)^{-1/8}$.

Realistic values of measures

- Taagepera (1972) claims that parliament size grows as cube root of population. This fits real data well, and we use it.
- Taagepera (2007) has a model which yields the following estimates, where M is the mean district magnitude and S the number of legislative seats:
  - The number of vote-getting parties is about $(MS)^{1/4}$.
  - L is about $(MS)^{1/6}$.
  - The maximum vote share of a party is about $(MS)^{-1/8}$.
- Li and Shugart (2016) have shown these formulae to work remarkably well on empirical data. We can use them to see that our simulated values scale appropriately.

Extensions — multiple districts

- Under the current model a party with zero support overall will remain with zero support. We can try to model formation of new parties.

Extensions — multiple districts

- Under the current model a party with zero support overall will remain with zero support. We can try to model formation of new parties.
- Instead of choosing another district randomly to imitate, make it more likely to imitate a "close" district.

Extensions — multiple districts

- Under the current model a party with zero support overall will remain with zero support. We can try to model formation of new parties.
- Instead of choosing another district randomly to imitate, make it more likely to imitate a "close" district.
- With enough tuning, it may be possible to use this model to describe preference change in real situations and even to use it for forecasting.

Representation versus stability

- It is widely believed that there is some sort of tradeoff between these criteria for an electoral system. At least at the extremes (dictatorship, direct democracy), they seem to be in conflict.

Representation versus stability

- It is widely believed that there is some sort of tradeoff between these criteria for an electoral system. At least at the extremes (dictatorship, direct democracy), they seem to be in conflict.
- It is possible *a priori* that the two criteria can be co-optimized only at the extremes, or somewhere in between the extremes.

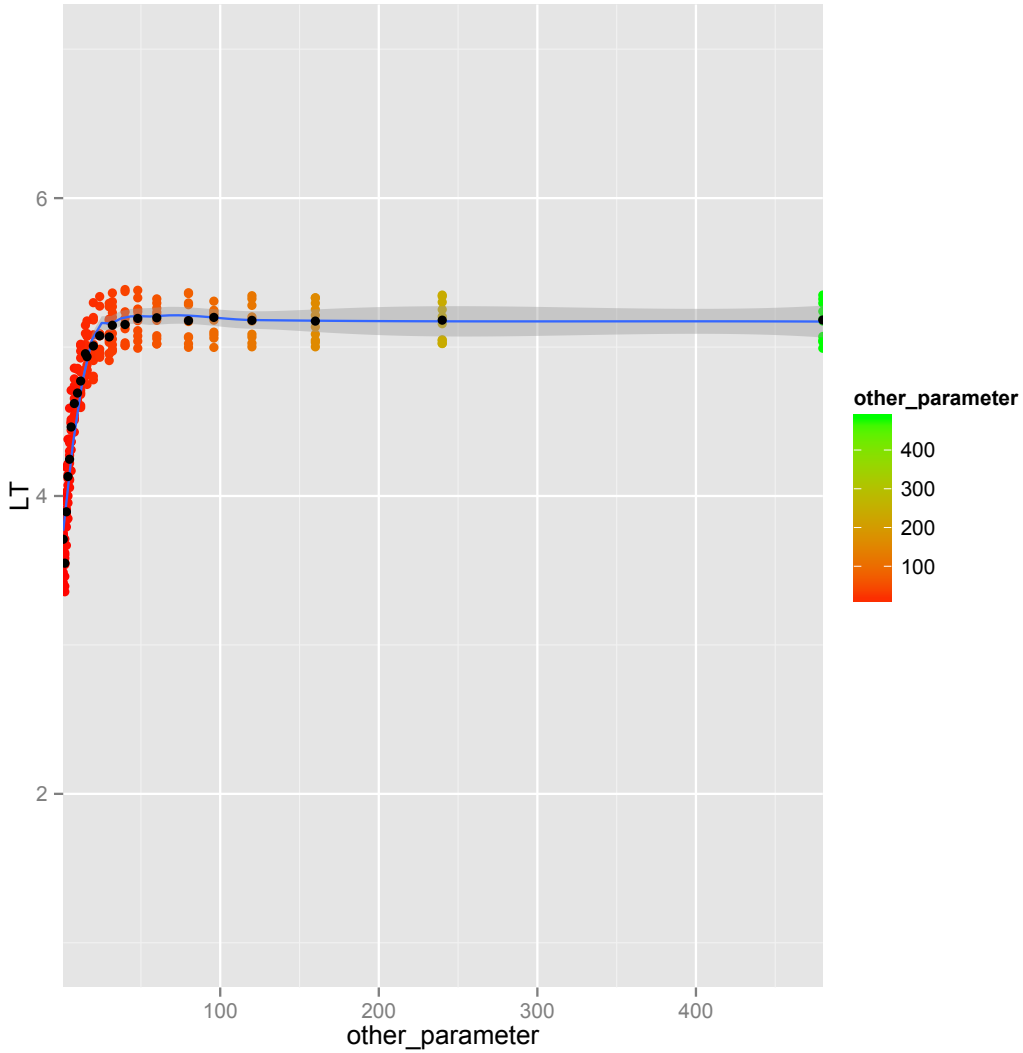
Representation versus stability

- It is widely believed that there is some sort of tradeoff between these criteria for an electoral system. At least at the extremes (dictatorship, direct democracy), they seem to be in conflict.
- It is possible *a priori* that the two criteria can be co-optimized only at the extremes, or somewhere in between the extremes.
- Carey & Hix ("The Electoral Sweet Spot", 2010) investigated whether seat-vote proportionality and party fragmentation can be co-optimized only at the extremes, or whether we can do better.

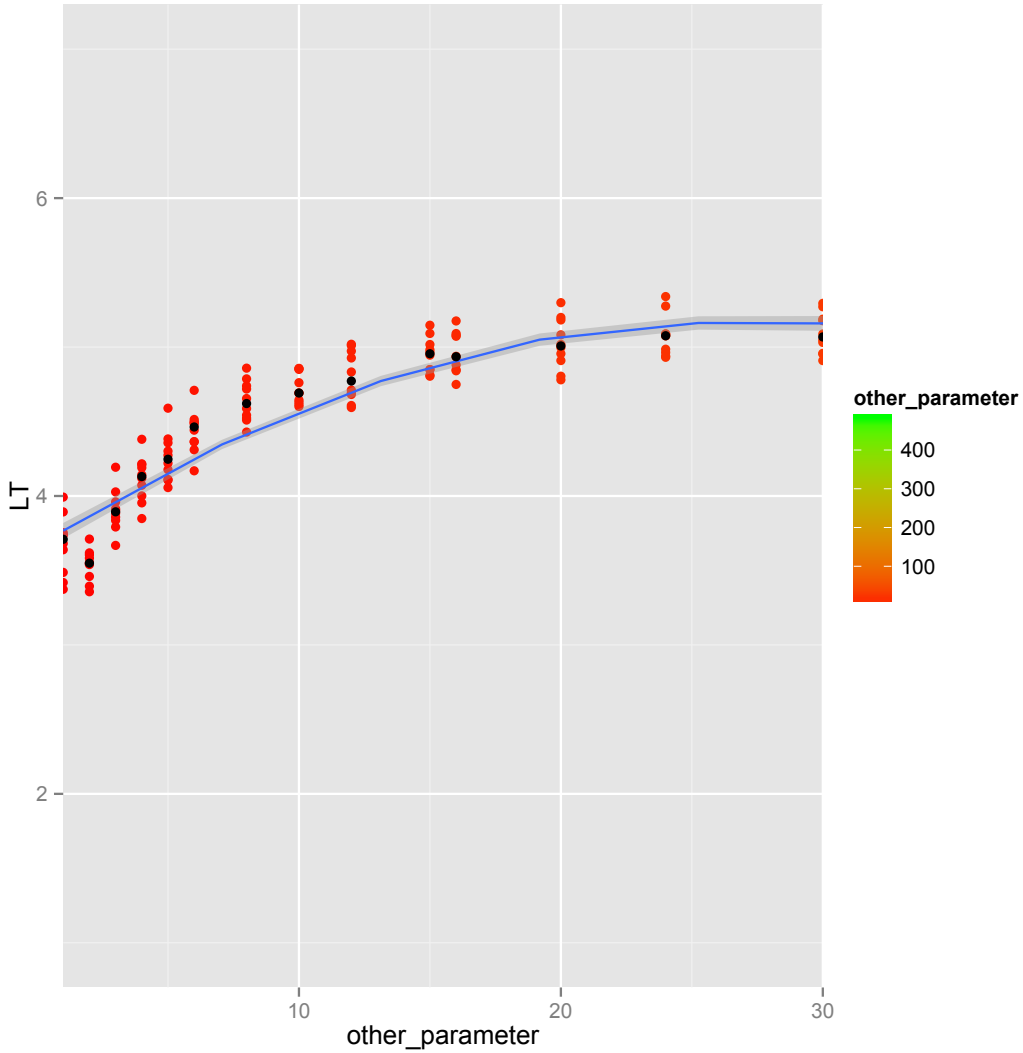
Representation versus stability

- It is widely believed that there is some sort of tradeoff between these criteria for an electoral system. At least at the extremes (dictatorship, direct democracy), they seem to be in conflict.
- It is possible *a priori* that the two criteria can be co-optimized only at the extremes, or somewhere in between the extremes.
- Carey & Hix ("The Electoral Sweet Spot", 2010) investigated whether seat-vote proportionality and party fragmentation can be co-optimized only at the extremes, or whether we can do better.
- They studied 609 election outcomes from 81 countries during 1945–2006. They conclude that low to moderate (say 3–7) district magnitude achieves the best tradeoff.

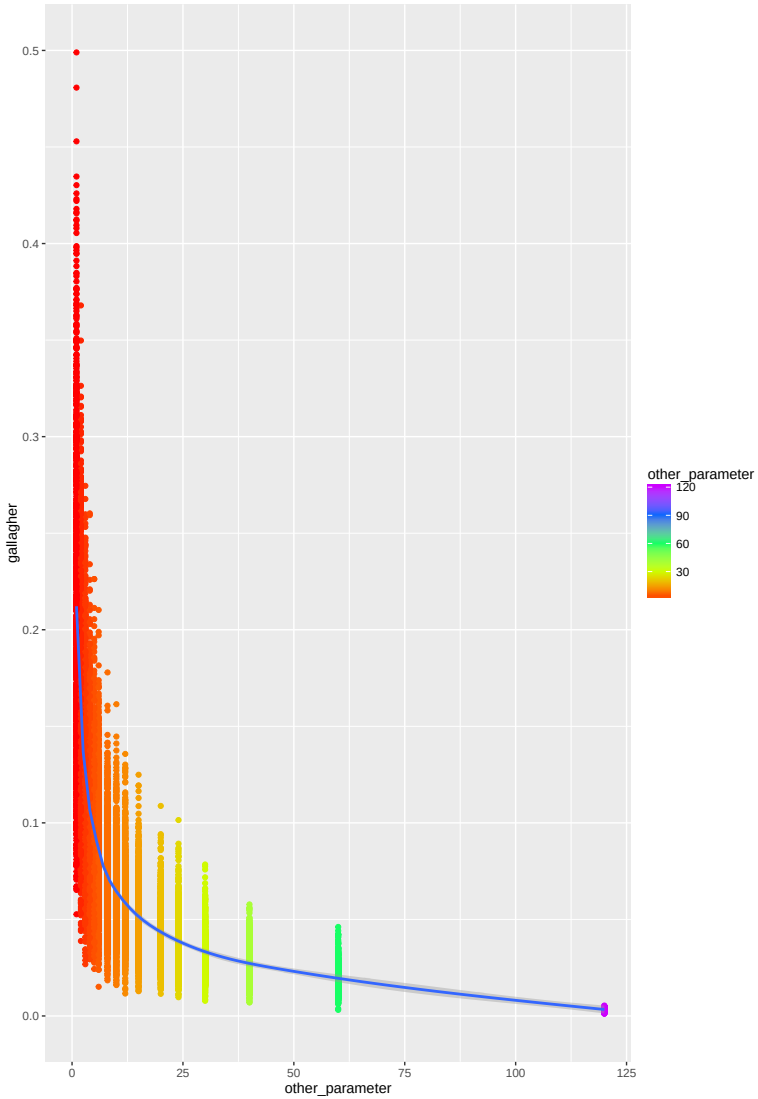
Number of parties versus district magnitude — simple



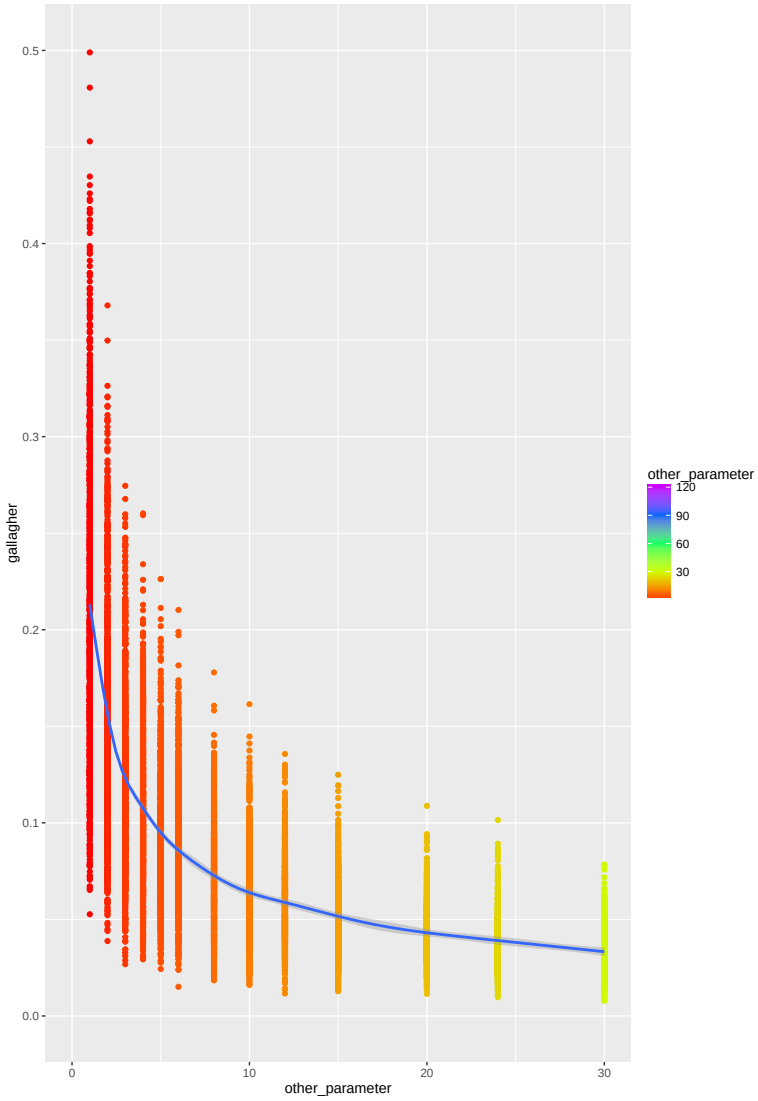
Number of parties versus district magnitude — simple



Disproportionality versus district magnitude — simple



Disproportionality versus district magnitude — simple



Number of parties versus disproportionality — simple