Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior

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Dartmouth College 2020-02-04

Example

- How many n-step lattice walks are there, if walks start from the origin, are confined to the first quadrant, and take steps in $\{S, NE, NW\}$? Call this a_n.
- Now reverse the steps to get $\{N, SE, SW\}$; call the analogous quantity b_n.
- Conjectured by Bostan & Kauers (2009):

$$a_n \sim 3^n \sqrt{\frac{3}{4\pi n}}$$

$$b_n \sim (2\sqrt{2})^n \frac{\theta(n)}{\pi n^2}$$

$$\theta(n) = \begin{cases} 24\sqrt{2} & \text{if } n \text{ is even} \\ 32 & \text{if } n \text{ is odd.} \end{cases}$$

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Overview — walks

- Consider nearest-neighbour walks in $\mathbb{Z}^2$, defined by a set $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{\mathbf{0}\}$ of short steps.
- We can consider restrictions, e.g. halfspace, nonnegative quadrant, return to x or y-axis, return to the origin.
- We keep track of the endpoint, and also the length. This gives a trivariate sequence $a_{r,s,n}$ with generating function (GF)

$$C(x, y, t) := \sum_{r,s,n} a_{r,s,n} x^r y^s t^n.$$

- Summing over r, s gives a univariate series $C(1, 1, t) := f(t) = \sum_n f_n t^n$.
- We seek in particular the asymptotics of f_n.

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Interlude - a hierarchy of generating functions

- Rational functions — constant coefficient linear recurrence for coefficients. Example: Fibonacci numbers.
- Algebraic functions — diagonals of rational functions. Example: Catalan numbers.
- D-finite functions (satisfy linear ODE with polynomial coefficients) — polynomial coefficient linear recurrence for coefficients. Example: $\binom{3n}{n}$, Bessel functions.
- Differentially algebraic functions (nonlinear ODE). Example: Bell numbers (egf).
- Worse! Differentially transcendental. Example: $\Gamma(z)$, Bell numbers (ogf).

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A hierarchy of generating functions from lattice walks

- Unrestricted walks — rational functions — have been understood "forever".
- Walks confined to a halfspace — algebraic functions — understood since Bousquet-Mélou & Petkovšek (2000), using the *kernel method*.
- 23 classes of walks confined to a quadrant — D-finite functions — reasonably well understood.
- 56 quadrant classes, steps that are not small — non D-finite functions — poorly understood.

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Previous work on walks in the quadrant, I

- Bousquet-Mélou & Mishna (2010): there are 79 inequivalent nontrivial cases.
- They introduced a symmetry group $G(\mathcal{S})$ and showed that it is finite in exactly 23 cases.
- They used finiteness to show for 22 cases that $C(x, y, t)$ is D-finite. For 19 of these, used the orbit sum method and for 3 more, the half orbit sum method.
- Bostan & Kauers (2010) explicitly showed that for the 23rd case (Gessel walks), $f(t)$ is algebraic (and hence D-finite).
- In the other 56 cases, $f(t)$ is indeed apparently not D-finite. So there are 23 nice inequivalent cases to discuss now.

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- Bostan, Chyzak, van Hoeij, Kauers & Pech (2016): expressed $f(t)$ in terms of hypergeometric integrals in 19 of these cases.
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- Open: proof of asymptotics of f_n for 15 cases. We solve that here via a unified approach.

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Standing assumptions

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The generating function (GF) is the formal power series

$$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}.$$

- Assume $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ where G, H are polynomials. The singular variety $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$ consists of poles.
- To avoid discussing complicated topology, assume all coefficients of F are nonnegative.

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Outline of ACSV project results (steps 3–5)

- Given a direction $\bar{\mathbf{r}}$, to compute asymptotics of $a_{\mathbf{r}}$ in that direction we first restrict to a variety $\mathrm{crit}(\bar{\mathbf{r}})$ of critical points.
- A subset $\mathrm{contrib}(\bar{\mathbf{r}}) \subseteq \mathrm{crit}(\bar{\mathbf{r}})$ contributes to asymptotics.
- For $\mathbf{p} \in \mathrm{contrib}(\bar{\mathbf{r}})$, there is a full asymptotic series $\mathcal{A}(\mathbf{p})$ depending on the type of singularity at $\mathbf{p}$. Each term is computable from finitely many derivatives of G and H at $\mathbf{p}$.
- This yields an asymptotic expansion

$$a_{\mathbf{r}} \sim \sum_{\mathbf{p} \in \mathrm{contrib}(\bar{\mathbf{r}})} \mathbf{p}^{-\mathbf{r}} \mathcal{A}(\mathbf{p})$$

that is uniform on compact subsets of directions, provided the geometry at $\mathbf{p}$ does not change.

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Smooth formulae for general d

- $\mathbf{z}_*$ turns out to be a critical point for $\overline{\mathbf{r}}$ iff the outward normal to $\log \mathcal{V}$ is parallel to $\mathbf{r}$. In other words, for some $\lambda \in \mathbb{C}$, $\mathbf{z}_*$ solves

$$\nabla_{\log} H(\mathbf{z}) := (z_1 H_1, \ldots, z_d H_d) = \lambda \mathbf{r}, H(\mathbf{z}) = \mathbf{0}.$$

- $$a_{\mathbf{r}} \sim \mathbf{z}_*(\overline{\mathbf{r}})^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2}\kappa(\mathbf{z}_*)}} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}$$

  where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.
- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.

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Outline of approach

- Can in fact get results for weighted walks in general dimension.
- Express f_n as diagonal coefficients of $d+1$-variable rational GF F, using the kernel method, orbit sum method, and series manipulations.
- Use mvGF theory of Pemantle and Wilson to extract asymptotics.
- Difficulty 1: singular set of F causes problems and F may have nonpositive coefficients.
- Difficulty 2: numerator often vanishes at points contributing to asymptotics, making general formulae hard to derive.
- Solution 1: ask Mireille Bousquet-Mélou!
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General dimension

- Melczer-Mishna analysed the case where $\mathcal{S}$ is symmetric over all d axes.
- We analyse the case with $d-1$ axes of symmetry (with weights having the same symmetry).
- Examples show that with fewer than $d-1$ symmetries, the GF is not D-finite, so such an approach must fail.
- We write $S(\mathbf{z}) = \sum_{\mathbf{i}\in\mathcal{S}} w_{\mathbf{i}}\mathbf{z}^{\mathbf{i}} = z_d B + Q + \overline{z}_d A$ where $\overline{z} = z^{-1}$ and A, B, Q are independent of z_d.
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Theorem

$$F(\mathbf{1}, t) = \Delta\left(\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)}\right),$$

where

$$G(\mathbf{z}, t) = (1 + z_1) \cdots (1 + z_{d-1})\,(1 - t z_1 \cdots z_d\,(Q + 2 z_d A))$$
$$H(\mathbf{z}, t) = (1 - z_d)\Big(1 - t z_1 \cdots z_d \overline{S}(\mathbf{z})\Big)\Big(1 - t z_1 \cdots z_d\,(Q + z_d A)\Big),$$

and

$$\overline{S}(\mathbf{z}) = S(\mathbf{z}_{\hat{d}}, \overline{z}_d) = \overline{z}_d B\left(\mathbf{z}_{\hat{d}}\right) + Q\left(\mathbf{z}_{\hat{d}}\right) + z_d A\left(\mathbf{z}_{\hat{d}}\right).$$

Theorem (Positive Drift Asymptotics)

Let

$$b_k = \sum_{\mathbf{i}\in\mathcal{S}, i_k=1} w_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathcal{S}, i_k=-1} w_{\mathbf{i}}.$$

for $1 \leq k < d$. *Then*

$$f_n \sim S(\mathbf{1})^n \cdot n^{\frac{-(d-1)}{2}} \cdot \left[\left(1 - \frac{A(\mathbf{1})}{B(\mathbf{1})}\right)\left(\frac{S(\mathbf{1})}{\pi}\right)^{\frac{d-1}{2}} \frac{1}{\sqrt{b_1 \cdots b_{d-1}}}\right].$$

Theorem (Negative Drift Asymptotics)

Let $\rho = \sqrt{\frac{A(\mathbf{1})}{B(\mathbf{1})}}$, *let* $b_k(\mathbf{z}_{\hat{k}}) := [z_k]S(\mathbf{z}) = [z_k^{-1}]S(\mathbf{z})$ *and let*

$$C_\rho := \frac{S(\mathbf{1},\rho)\,\rho}{2\,\pi^{d/2}\,A(\mathbf{1})(1-1/\rho)^2} \cdot \sqrt{\frac{S(\mathbf{1},\rho)^d}{\rho\, b_1(\mathbf{1},\rho)\cdots b_{d-1}(\mathbf{1},\rho)\cdot B(\mathbf{1})}}.$$

- *If* $Q \neq 0$ *then*

$$f_n \sim S(\mathbf{1},\rho)^n \cdot n^{-d/2-1} \cdot C_\rho.$$

- *If* $Q = 0$ *then*

$$f_n \sim n^{-d/2-1} \cdot \Big[S(\mathbf{1},\rho)^n \cdot C_\rho + S(\mathbf{1},-\rho)^n \cdot C_{-\rho} \Big].$$

Example

Consider the model defined by $\mathcal{S} = \{(1,0),(-1,0),(0,1),(0,-1)\}$, where the south step $(0,-1)$ has weight $a > 0$ and the north step $(0,1)$ has weight $b > 0$ (when a and b are integers we can think of having multiple copies of each step with different colours). Then

$$A(x) = a \qquad Q(x) = \overline{x} + x \qquad B(x) = b$$

and

$$f_n \sim \begin{cases} \left(2+2\sqrt{ab}\right)^n \cdot n^{-2} \cdot \frac{2a^{1/4}\left(1+\sqrt{ab}\right)^2}{\pi b^{3/4}\left(\sqrt{a}-\sqrt{b}\right)^2} & : b < a \\ (2+2a)^n \cdot n^{-1} \cdot \frac{2(1+a)}{\sqrt{a}\,\pi} & : b = a \\ (2+a+b)^n \cdot n^{-1/2} \cdot \frac{(a+b)\sqrt{2+a+b}}{b\sqrt{\pi}} & : b > a \end{cases}$$

with the different cases corresponding to negative drift, zero drift, and positive drift.

$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics
	$\frac{4}{\pi} \cdot \frac{4^n}{n}$		$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}$		$\frac{A_n}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{2}{\pi} \cdot \frac{4^n}{n}$		$\frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}}$		$\frac{B_n}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$
	$\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$		$\frac{\sqrt{5}}{3\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{C_n}{\pi} \cdot \frac{(2\sqrt{6})^n}{n^2}$
	$\frac{8}{3\pi} \cdot \frac{8^n}{n}$		$\frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2}$
	$\frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}}$		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2}$
	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$		$\frac{\sqrt{7}}{3\sqrt{3\pi}} \cdot \frac{7^n}{\sqrt{n}}$		$\frac{\sqrt{570-114\sqrt{6}}(24\sqrt{6}+59)}{19\pi} \cdot \frac{(2+2\sqrt{6})^n}{n^2}$
	$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$		$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$
	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{2/3}}$		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}}$		

Table: Asymptotics for the 23 D-finite models.

$$A_n = \begin{cases} 24\sqrt{2} & : n \text{ even} \\ 32 & : n \text{ odd} \end{cases}, \quad B_n = \begin{cases} 12\sqrt{3} & : n \text{ even} \\ 18 & : n \text{ odd} \end{cases}, \quad C_n = \begin{cases} 12\sqrt{30} & : n \text{ even} \\ 144/\sqrt{5} & : n \text{ odd} \end{cases}$$

Extensions

- Small modifications yield results for walks constrained to return to an axis or the origin.
- Walks in Weyl chambers can be treated in this way.
- The zero-drift case is tricky; we worked out the generic case but there are many non-generic subcases.

Part of table of results for excursions

$\mathcal{S}$	Return to x-axis	Return to y-axis	Return to origin
	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$
	$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{8}{\pi} \cdot \frac{4^n}{n^3}$
	$\frac{3\sqrt{6}}{2\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{2\sqrt{6}}{\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{3\sqrt{6}}{\pi} \cdot \frac{6^n}{n^3}$
	$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{128}{27\pi} \cdot \frac{8^n}{n^3}$
	$\frac{3\sqrt{3}}{4\sqrt{\pi}} \frac{3^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$	$\epsilon_n \frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3}$
	$\frac{8}{3\sqrt{\pi}} \frac{4^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	$\delta_n \frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^3}$
	$\frac{5\sqrt{10}}{16\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\frac{\sqrt{2}(1+\sqrt{2})^{3/2}}{\pi} \frac{(2+2\sqrt{2})^n}{n^2}$	$\frac{2(1+\sqrt{2})^{3/2}}{\pi} \frac{(2+2\sqrt{2})^n}{n^3}$
	$\frac{5\sqrt{10}}{24\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{30}}{5\pi} \frac{(2\sqrt{6})^n}{n^2}$	$\delta_n \frac{24\sqrt{30}}{25\pi} \frac{(2\sqrt{6})^n}{n^3}$

Deriving generating function: kernel method

- Introduced by Knuth and developed by Bousquet-Mélou and others into a powerful tool.
- Recursion gives

$$(1 - tS(\mathbf{z}))z_1 \cdots z_d F(\mathbf{z}, t) = z_1 \cdots z_d + \sum_{k=1}^{d} L_k(\mathbf{z}_{\hat{k}}, t)$$

  where $\qquad L_k(\mathbf{z}_{\hat{k}}, t) \in \mathbb{Q}[\mathbf{z}_{\hat{k}}][[t]].$
- There is a symmetry group of S generated by maps $z_k \mapsto 1/z_k$ and $z_d \mapsto \overline{z}_d \frac{A(\mathbf{z}_{\hat{d}})}{B(\mathbf{z}_{\hat{d}})}$.
- An alternating sum over the group almost fixes the left side and kills the L_k terms on the right, allowing us to solve for the power series F by taking the terms with no negative powers.
- We use a simple change of variable to convert the positive part of a Laurent series to the diagonal of a series.

Hörmander's explicit formula

The asymptotic contribution of an isolated nondegenerate stationary point is

$$\left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where L_k is a differential operator of order $2k$ evaluated at $\mathbf{0}$. Specifically,

$$\begin{aligned} \underline{f}(t) &= f(t) - (1/2)tf''(0)t^T \\ \mathcal{D} &= \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b}(-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b) \\ L_k(A, f) &= \sum_{l\leq 2k} \frac{\mathcal{D}^{l+k}(A\underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}. \end{aligned}$$

Example (2-D case with no symmetry: $\mathcal{S} = \{N, W, SE\}$)

It turns out that

$$F(t) = \Delta\left(\frac{(x^2 - y)(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))}\right).$$

We decompose

$$\frac{(x^2 - y)(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))} = -\frac{(1 - \overline{xy})(x - y^2)(x + 1)}{(1 - y)(1 - xyt(\overline{y} + y\overline{x} + x))} + \frac{(1 - \overline{xy})(x - y^2)}{(1 - x)(1 - xyt(\overline{y} + y\overline{x} + x))},$$

Our usual methods now yield

$$f_n = \frac{3^n}{n^{3/2}}\left(\frac{3\sqrt{3}}{2\sqrt{\pi}} + O\left(n^{-1}\right)\right).$$

References

- S. Melczer & M. C. Wilson, *Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior*. http://arxiv.org/abs/1810.06170.
- R. Pemantle & M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. http://ACSVproject.org
- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.

Publication reform

- Pressure is building for complete conversion of the journal system to open access (e.g. Plan S from European research funders)
- Large commercial publishers have incentives not aligned with scholarship or the interests of readers and authors, and provide overall low quality service for very high prices.
- The journal market is dysfunctional (not properly competitive).
- I am associated with several organizations aiming to improve this: MathOA, Free Journal Network, Publishing Reform Forum. If you would like to help or learn more, please contact me.