# Lattice path asymptotics via Analytic Combinatorics in Several Variables

Mark C. Wilson
Department of Computer Science
University of Auckland

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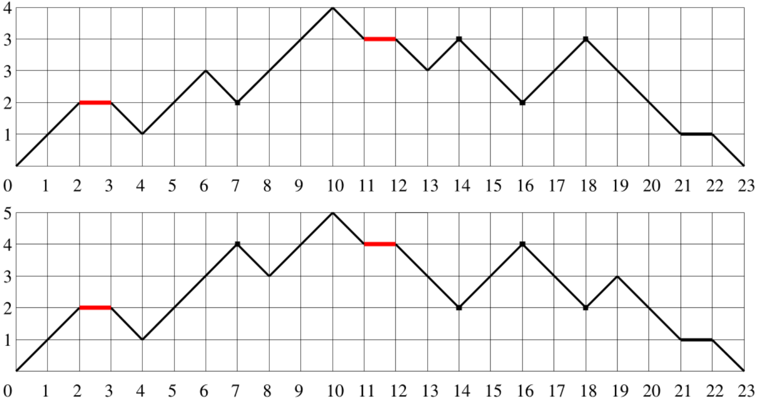
# Some lattice walks




Lattice walks have many applications: modelling physical and chemical structures, encoding trees, statistical inference. Their random analogues are important in queueing theory.

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- This has both forward and back steps in each dimension, and it is not so easy to derive an explicit formula.
- Conjectured by Bostan & Kauers:

$$a_n \sim 3^n \sqrt{\frac{3}{4\pi n}}.$$

# Overview — walks

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- We keep track of the endpoint, and also the length. This gives a trivariate sequence $a_{r,s,n}$ with generating function (GF)

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- Summing over $r, s$ gives a univariate series $C(1, 1, t) := f(t) = \sum_n f_n t^n$.
- We seek in particular the asymptotics of $f_n$.

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- 23 classes of walks confined to a quadrant — D-finite functions — satisfy a linear ODE with polynomial coefficients — reasonably well understood.
- 56 quadrant classes, steps that are not small — non D-finite functions — poorly understood.

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- Bostan & Kauers (2010) explicitly showed that for the 23rd case (Gessel walks), $f(t)$ is algebraic (and hence D-finite).
- In the other 56 cases, $f(t)$ is indeed not D-finite. So there are 23 nice inequivalent cases to discuss now.

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- Open: proof of asymptotics of $f_n$ for 15 cases. We solve that here via a unified approach.

## Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

| | OEIS | $\mathfrak{S}$ | alg | equiv | | OEIS | $\mathfrak{S}$ | alg | equiv |
|---|---|---|---|---|---|---|---|---|---|
| 1 | A005566 |  | N | $\frac{4}{\pi}\frac{4^n}{n}$ | 13 | A151275 |  | N | $\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$ |
| 2 | A018224 |  | N | $\frac{2}{\pi}\frac{4^n}{n}$ | 14 | A151314 |  | N | $\frac{\sqrt{6\lambda\mu}C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$ |
| 3 | A151312 |  | N | $\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$ | 15 | A151255 |  | N | $\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$ |
| 4 | A151331 |  | N | $\frac{8}{3\pi}\frac{8^n}{n}$ | 16 | A151287 |  | N | $\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$ |
| 5 | A151266 |  | N | $\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$ | 17 | A001006 |  | Y | $\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$ |
| 6 | A151307 |  | N | $\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$ | 18 | A129400 |  | Y | $\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$ |
| 7 | A151291 |  | N | $\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$ | 19 | A005558 |  | N | $\frac{8}{\pi}\frac{4^n}{n^2}$ |
| 8 | A151326 |  | N | $\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$ | | | | | |
| 9 | A151302 |  | N | $\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$ | 20 | A151265 |  | Y | $\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$ |
| 10 | A151329 |  | N | $\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$ | 21 | A151278 |  | Y | $\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$ |
| 11 | A151261 |  | N | $\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$ | 22 | A151323 |  | Y | $\frac{\sqrt{2}\,3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$ |
| 12 | A151297 |  | N | $\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$ | 23 | A060900 |  | Y | $\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$ |

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

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- Cons: $f(t)$ may be nasty to describe (not algebraic). We don't need to compute it if we only want the asymptotics of its coefficients. The computational effort can be enormous.
- Our idea: there is a tradeoff between niceness of generating function and dimension. We use recently developed tools for asymptotics of higher dimensional rational GFs.

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- Another approach uses hypergeometric integrals. This requires computation of integrals which have not yet been done explicitly.

# Diagonals

- The orbit sum approach yields $f$ as the positive part of a rational series. This is the leading diagonal of a closely related series. Thus we have $f = \operatorname{diag} F$ where

$$F(x, y, t) = \frac{xyP(x^{-1}, y^{-1})}{(1 - txyS(x^{-1}, y^{-1}))\,(1 - x)(1 - y)}$$

and $S$ and $P$ are Laurent polynomials:

$$S(x, y) = \sum_{(i,j) \in S} x^i y^j \text{ (step enumerator )}$$

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- The trivariate GF is rational but the diagonal is only D-finite and can't be easily described. We instead compute asymptotics of $[x^n y^n t^n] f(x,y,t)$.

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- The ultimate justification involves Morse theory, but convex analysis often suffices in the combinatorial case.
- We deal in particular with multiple points (locally a transverse intersection of $k$ smooth factors). If $1 \leq k \leq d$, formulae look like

$$a_{\mathbf{r}} \sim {\mathbf{z}_*}^{-\mathbf{r}} \sum_{l \geq 0} b_l ||\mathbf{r}||^{-(d-k)/2-l}.$$

## Example (Univariate pole: Fibonacci)

- Consider $F(z) = z/(1 - z - z^2)$, the GF for Fibonacci numbers. There are two poles, at $\phi := 2/(1 + \sqrt{5})$ and $-\phi^{-1}$. Using a circle of radius $\phi - \varepsilon$ yields, by Cauchy's theorem

$$a_r = \frac{1}{2\pi i} \int_{C_{\phi-\varepsilon}} z^{-r-1} F(z) \, dz$$

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- The integral is $O((\phi + \varepsilon)^{-r})$ while the residue is order $\phi^{-r}/\sqrt{5}$. Thus $[z^r]F(z) \sim \phi^{-r}/\sqrt{5}$ as $r \to \infty$.

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- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over $R$. In this example, $R = n$ is the minimum.
- The integral over $C_n$ has most mass near $z = n$, so that

$$
\begin{aligned}
a_n &= \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)} \, d\theta \\
&\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right) \, d\theta.
\end{aligned}
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## Example (Saddle point example continued)

- The Maclaurin expansion yields

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- This gives, with $b_n = 2\pi n^n e^{-n} a_n$, Laplace's approximation:

$$b_n \approx \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2)\, d\theta \approx \int_{-\infty}^{\infty} \exp(-n\theta^2/2)\, d\theta = \sqrt{2\pi/n}.$$

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- This recaptures Stirling's approximation, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}.$$

# Multivariate asymptotics — some quotations

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- (Flajolet/Sedgewick 2009) "Roughly, we regard here a bivariate GF as a collection of univariate GFs . . . ."
- We aimed to improve the multivariate situation.

# Outline of results (generic case)

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- There is an expansion $a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$ where $\mathrm{formula}(\mathbf{z}_*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of $G$ and $H$ at $\mathbf{z}_*$.

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- This yields

$$a_{\mathbf{r}} \sim \mathrm{formula}(\mathbf{z}_*)$$

  where the expansion is uniform on compact subsets of directions, provided the geometry does not change.

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- Initial terms can vanish in expansion, higher ones harder to compute explicitly.
- Geometry of $\mathcal{V}$ near $\mathbf{z}_*$ can be strange.
- All of these occur in applications, the first three in our lattice point analysis.

# Results for multiple points: generic shape of $\mathrm{formula}(\mathbf{z}_*)$

- (smooth point, or multiple point with $n \le d$)

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  - (smooth/multiple point $n < d$)

$$a_0 = G(\mathbf{z}_*)C(\mathbf{z}_*)$$

    where $C$ depends on the derivatives to order $2$ of $H$;

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$$\mathbf{z}_*{}^{-\mathbf{r}} \sum_{k \geq 0} a_k |\mathbf{r}|^{-(d-n)/2-k}.$$

  - (smooth/multiple point $n < d$)

  $$a_0 = G(\mathbf{z}_*) C(\mathbf{z}_*)$$

  where $C$ depends on the derivatives to order 2 of $H$;
  - (multiple point, $n = d$)

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$$\mathbf{z}_*{}^{-\mathbf{r}} G(\mathbf{z}_*) P\left(\frac{r_1}{z_1^*}, \ldots, \frac{r_d}{z_d^*}\right),$$

$P$ a piecewise polynomial of degree $n - d$.

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- Cons: non-generic behaviour occurs.
- In the following, we use notation

$$S_j = \{i : (i, j) \in S\} \qquad \text{for each } j \in \{-1, 0, 1\}.$$

# Singularities

- The factor $H_1 := 1 - txyS(x^{-1}, y^{-1})$ is a polynomial. Then

$$\begin{aligned}\nabla_{\log} H_1 &:= (x\partial H_1/\partial x, y\partial H_1/\partial y, t\partial H_1/\partial t)\\ &= (-1 + ty\partial S/\partial x, -1 + tx\partial S/\partial y, -1)\end{aligned}$$

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  and thus this factor is everywhere smooth.
- Other singularities come from factors of $(1-x), (1-y)$ and possibly from clearing denominators of $xyP(x^{-1}, y^{-1})$.
- When $F$ is combinatorial, there is a dominant singularity for direction $\mathbf{1}$ lying in the positive orthant.

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- If $S$ has a vertical axis of symmetry, then $(x^2 - 1) \sum_j y^j = 0$.

# Structure of $G$

- Write

$$\begin{aligned} S(x,y) &= y^{-1}A_{-1}(x) + A_0(x) + yA_1(x) \\ &= x^{-1}B_{-1}(y) + B_0(y) + xB_1(y). \end{aligned}$$

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- $G$ is dihedral, generated by the involutions (considered as algebra homomorphisms)

$$(x,y) \mapsto \left(x^{-1}\frac{B_{-1}(y)}{B_1(y)}, y\right)$$
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- If $S$ has vertical symmetry then $B_1 = B_{-1}$, these maps commute, and $G$ has order $4$.

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- This covers Cases 1–16. The possible denominators from $P$ are $x^2 + 1, x^2 + x + 1$. Neither can contribute because the problem is combinatorial and aperiodic. The dominant point has $x = 1$.

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- Thus for Cases 5–10 we have leading term $C|S|^n n^{-1}$.

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- This happens in all cases 11–16. The numerator simplifies at the smooth point to $(1+x)(1-y^2|S_{-1}|/|S_1|)$, which is zero from the critical point equation for $y$.
- The leading term asymptotic is $C(|S_0| + 2\sqrt{|S_1||S_{-1}|})^n n^{-2}$.

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- If this is positive, there are more possible walks that don't cross the boundary, so the quarter plane restriction is encountered less often. Asymptotics come from the double point $(1, 1, 1/|S|)$.
- If the drift is nonpositive, asymptotics come from the highest smooth point.
- This explains Cases 1–16 in a unified way. We could derive higher order asymptotics too (e.g. using Sage package implementing Raichev & Wilson papers).

# Other cases

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- Cases 17–19 also follow as above, with slightly different formulae and more work.
- Cases 20–23 are harder. We don't have a nice diagonal expression, and the conjectured asymptotics show that analysis will be trickier. However the GFs are known to be algebraic and 1-dimensional methods can be used.

# Extensions

- We can derive similar expressions for the number of walks returning to the $x$-axis, the $y$-axis, or the origin. A very similar analysis proves recently conjectured asymptotics of Bostan, Chyzak, van Hoeij, Kauers, and Pech.

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- Usually, the asymptotics are changed by a factor of $n$ or $\sqrt{n}$. Sometimes the exponential rate changes, depending on the shape of the step set.
- Our approach allows for unified analysis of rational trivariate GFs, which provides results and insight, rather than ad hoc analysis of complicated univariate GFs, which provides results sometimes and no insight.

| $n$ | $\mathcal{S}$ | Asymptotics | $n$ | $\mathcal{S}$ | Asymptotics | $n$ | $\mathcal{S}$ | Asymptotics |
|---|---|---|---|---|---|---|---|---|
| 1 |  | $\frac{4}{\pi}\cdot\frac{4^n}{n}$ | 9 |  | $\frac{\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{3^n}{\sqrt{n}}$ | 17 |  | $\frac{4\cdot A_n}{\pi}\cdot\frac{(2\sqrt{2})^n}{n^2}$ |
| 2 |  | $\frac{2}{\pi}\cdot\frac{4^n}{n}$ | 10 |  | $\frac{4}{3\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}$ | 18 |  | $\frac{3\sqrt{3}\cdot B_n}{\pi}\cdot\frac{(2\sqrt{3})^n}{n^2}$ |
| 3 |  | $\frac{\sqrt{6}}{\pi}\cdot\frac{6^n}{n}$ | 11 |  | $\frac{\sqrt{5}}{2\sqrt{2\pi}}\cdot\frac{5^n}{\sqrt{n}}$ | 19 |  | $\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi}\cdot\frac{(2+2\sqrt{2})^n}{n^2}$ |
| 4 |  | $\frac{8}{3\pi}\cdot\frac{8^n}{n}$ | 12 |  | $\frac{\sqrt{5}}{3\sqrt{2\pi}}\cdot\frac{5^n}{\sqrt{n}}$ | 20 |  | $\frac{6C_n}{\pi}\cdot\frac{(2\sqrt{6})^n}{n^2}$ |
| 5 |  | $\frac{2\sqrt{2}}{\Gamma(1/4)}\cdot\frac{3^n}{n^{3/4}}$ | 13 |  | $\frac{2\sqrt{3}}{3\sqrt{\pi}}\cdot\frac{6^n}{\sqrt{n}}$ | 21 |  | $\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi}\cdot\frac{(2+2\sqrt{3})^n}{n^2}$ |
| 6 |  | $\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\cdot\frac{3^n}{n^{3/4}}$ | 14 |  | $\frac{\sqrt{7}}{3\sqrt{3\pi}}\cdot\frac{7^n}{\sqrt{n}}$ | 22 |  | $\frac{\sqrt{6(379+156\sqrt{6})(1+\sqrt{6})^7}}{5\sqrt{95}\pi}\cdot\frac{(2+2\sqrt{6})^n}{n^2}$ |
| 7 |  | $\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)}\cdot\frac{6^n}{n^{3/4}}$ | 15 |  | $\frac{3\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{3^n}{n^{3/2}}$ | 23 |  | $\frac{8}{\pi}\cdot\frac{4^n}{n^2}$ |
| 8 |  | $\frac{4\sqrt{3}}{3\Gamma(1/3)}\cdot\frac{4^n}{n^{2/3}}$ | 16 |  | $\frac{3\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{6^n}{n^{3/2}}$ | | | |

TABLE 1. Asymptotics for the 23 D-finite models.

$A_n=4(1-(-1)^n)+3\sqrt{2}(1+(-1)^n)$, $B_n=\sqrt{3}(1-(-1)^n)+2(1+(-1)^n)$, $C_n=12/\sqrt{5}(1-(-1)^n)+\sqrt{30}(1+(-1)^n)$

| $\mathcal{S}$ | $C(0,1,t)$ | $C(1,0,t)$ | $C(0,0,t)$ | $\mathcal{S}$ | $C(0,1,t)$ | $C(1,0,t)$ | $C(0,0,t)$ |
|---|---|---|---|---|---|---|---|
|  | $\frac{8}{\pi} \cdot \frac{4^n}{n^2}$ | $\frac{8}{\pi} \cdot \frac{4^n}{n^2}$ | $\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$ |  | $\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$ | $\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$ | $\delta_n \frac{8}{\pi} \cdot \frac{4^n}{n^3}$ |
|  | $\frac{3\sqrt{6}}{2\pi} \cdot \frac{6^n}{n^2}$ | $\delta_n \frac{2\sqrt{6}}{\pi} \cdot \frac{6^n}{n^2}$ | $\delta_n \frac{3\sqrt{6}}{\pi} \cdot \frac{6^n}{n^3}$ |  | $\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$ | $\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$ | $\frac{128}{27\pi} \cdot \frac{8^n}{n^3}$ |
|  | $\frac{3\sqrt{3}}{4\sqrt{\pi}} \frac{3^n}{n^{3/2}}$ | $\delta_n \frac{4\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$ | $\epsilon_n \frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3}$ |  | $\frac{8}{3\sqrt{\pi}} \frac{4^n}{n^{3/2}}$ | $\delta_n \frac{4\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$ | $\delta_n \frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^3}$ |
|  | $\frac{5\sqrt{10}}{16\sqrt{\pi}} \frac{5^n}{n^{3/2}}$ | $\frac{\sqrt{2}A^{3/2}}{\pi} \frac{(2A)^n}{n^2}$ | $\frac{2A^{3/2}}{\pi} \frac{(2A)^n}{n^3}$ |  | $\frac{5\sqrt{10}}{24\sqrt{\pi}} \frac{5^n}{n^{3/2}}$ | $\delta_n \frac{4\sqrt{30}}{5\pi} \frac{(2\sqrt{6})^n}{n^2}$ | $\delta_n \frac{24\sqrt{30}}{25\pi} \frac{(2\sqrt{6})^n}{n^3}$ |
|  | $\frac{\sqrt{3}}{\sqrt{\pi}} \frac{6^n}{n^{3/2}}$ | $\frac{2\sqrt{3}B^{3/2}}{3\pi} \frac{(2B)^n}{n^2}$ | $\frac{2B^{3/2}}{\pi} \frac{(2B)^n}{n^3}$ |  | $\frac{7\sqrt{21}}{54\sqrt{\pi}} \frac{7^n}{n^{3/2}}$ | $\frac{D}{285\pi} \frac{(2C)^n}{n^2}$ | $\frac{2E}{1805\pi} \frac{(2C)^n}{n^3}$ |
|  | $\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{3^n}{n^{5/2}}$ | $\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{3^n}{n^{5/2}}$ | $\sigma_n \frac{81\sqrt{3}}{\pi} \cdot \frac{3^n}{n^4}$ |  | $\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$ | $\frac{32}{\pi} \cdot \frac{4^n}{n^3}$ | $\delta_n \frac{768}{\pi} \cdot \frac{4^n}{n^5}$ |
|  | $\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^{5/2}}$ | $\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^{5/2}}$ | $\frac{27\sqrt{3}}{\pi} \cdot \frac{6^n}{n^4}$ | | | | |

TABLE 2. Asymptotics of boundary returns for the highly symmetric, positive drift, and sporadic cases.

| $\mathcal{S}$ | $C(0,1,t)$ | $\mathcal{S}$ | $C(0,1,t)$ |
|---|---|---|---|
|  | $\left(\epsilon_n \frac{448\sqrt{2}}{9\pi} + \epsilon_{n-1} \frac{640}{9\pi} + \epsilon_{n-2} \frac{416\sqrt{2}}{9\pi} + \epsilon_{n-3} \frac{512}{9\pi}\right) \cdot \frac{(2\sqrt{2})^n}{n^3}$ |  | $\left(\delta_n \frac{36\sqrt{3}}{\pi} + \delta_{n-1} \frac{54}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^3}\right) \cdot \frac{(2\sqrt{3})^n}{n^3}$ |
|  | $\frac{4A^{7/2}}{\pi} \cdot \frac{(2A)^n}{n^3}$ |  | $\left(\delta_n \frac{72\sqrt{30}}{5\pi} + \delta_{n-1} \frac{864\sqrt{5}}{25\pi}\right) \cdot \frac{(2\sqrt{6})^n}{n^3}$ |
|  | $\frac{3B^{7/2}}{2\pi} \cdot \frac{(2B)^n}{n^3}$ |  | $\frac{6(4571+1856\sqrt{6})\sqrt{23-3\sqrt{6}}}{1805\pi} \cdot \frac{(2C)^n}{n^3}$ |

TABLE 3. Asymptotics of $C(0,1,t)$ for negative drift cases; other asymptotics of $\mathcal{S}$ are the same as those of $-\mathcal{S}$ above.

$A = 1+\sqrt{2}$, $B = 1+\sqrt{3}$, $C = 1+\sqrt{6}$, $D = (156+41\sqrt{6})\sqrt{23-3\sqrt{6}}$, $E = (583+138\sqrt{6})\sqrt{23-3\sqrt{6}}$

$\delta_n = 1$ if $n \equiv 0 \bmod 2$, $\sigma_n = 1$ if $n \equiv 0 \bmod 3$, and $\epsilon_n = 1$ if $n \equiv 0 \bmod 4$ – each is 0 otherwise

# Possible future work

- Higher dimensions: $d = 3$ has been studied empirically by Bostan, Bousquet-Mélou, Kauers & Melczer. The orbit sum method appears to work relatively rarely, however.

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- Random walk variants can be treated by simply scaling the variables by probabilities. We anticipate few changes to the overall analysis.
- Walks in a Weyl chamber (Gessel & Zeilberger) yield very similar generating functions, analysable in the same way.

# Closing remarks

- The methods in the ACSV book are still under-utilized by other researchers. This problem was a fairly straightforward application of general theory.

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- Many researchers in enumeration use extra ("catalytic") variables and then throw them away; they ought to keep them and use multivariate methods more often.

# General references

- S. Melczer & M. C. Wilson, *Asymptotics of lattice walks via analytic combinatorics in several variables*. http://arxiv.org/abs/1511.02527.

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- R. Pemantle & M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. Draft available on my website.
- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.

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- S. Melczer & M. C. Wilson, *Asymptotics of lattice walks via analytic combinatorics in several variables*. http://arxiv.org/abs/1511.02527.
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- Sage implementations by Alex Raichev: https://github.com/araichev/amgf.
- S. Melczer & M. Mishna, *Asymptotic Lattice Path Enumeration Using Diagonals*, 2014.

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