Asymptotics of multivariate generating functions

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Sequences

- In combinatorics and probability we very often encounter special sequences of numbers (usually rational or integer).
- Simple examples:
  - a_n = number of binary trees with n nodes
  - a_n = expected height of a random binary tree with n nodes
  - $a_{n,k}$ = number of binary trees with n nodes having k leaves
  - $a_{\mathbf{r}}$ = number of paths by a d-dimensional rook from the origin to $\mathbf{r} \in \mathbb{N}^d$.
- A (d-variate) sequence is just a function $\mathbb{N}^d \to \mathbb{C}$. We use subscript notation, $a_{\mathbf{r}} := a(\mathbf{r})$.

Generating functions

- The best all-round tool for studying $a_{\mathbf{r}}$ is a d-variate formal power series called the generating function

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}.$$

- This is analogous to the Fourier or Laplace transform, but it converts a discrete problem into a continuous one.
- A recurrence relation for $a_{\mathbf{r}}$ corresponds to a functional equation for F.
- Introduced by Euler in 1753 to count diagonals in polygons (*Catalan numbers*).

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- For many naturally occurring problems, much of the computation can be algorithmically implemented in a computer algebra system.
- Versatility: GFs yield recurrences, identities, congruences, unimodality results, asymptotics.

Concrete univariate example: Fibonacci numbers

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- Let $F(z) = \sum a_n z^n$, $\theta := (1 + \sqrt{5})/2$ (golden ratio). Then the recurrence above yields $(1 - z - z^2)F(z) = z$ and by partial fractions

$$F(z) = \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left[\frac{1}{1 - \theta z} - \frac{1}{1 + \theta^{-1} z} \right].$$

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- Asymptotically, the θ^n term dominates.

Fibonacci numbers: more details of finding the GF

$$
\begin{aligned}
F(z) &:= \sum_{n\geq 0} a_n z^n \\
a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 2 \\
a_n z^n &= a_{n-1} z^n + a_{n-2} z^n \qquad \text{for } n \geq 2 \\
\sum_{n\geq 2} a_n z^n &= \sum_{n\geq 2} a_{n-1} z^n + \sum_{n\geq 2} a_{n-2} z^n \\
\sum_{n\geq 2} a_n z^n &= z \sum_{n\geq 1} a_n z^n + z^2 \sum_{n\geq 0} a_n z^n \\
F(z) - a_0 - a_1 z &= z(F(z) - a_0) + z^2 F(z) \\
F(z) &= \frac{a_0 + a_1 z}{1 - z - z^2} = \frac{z}{1 - z - z^2}.
\end{aligned}
$$

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- The dominant pole of F is at ρ where $1/2 < \rho < 1$ (by Pringsheim's theorem). Using Rouché's theorem we can show it is a simple pole and all other poles have $|z| > 0.6$.
- This method works for any regular language, which covers vastly many problems involving sequences/strings/words and unconstrained lattice walks.

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- Thus the exponential rate is $1/\rho$, and

$$a_n \sim \frac{c(\rho)}{\rho(11\rho^{10} + (1 - 2\rho)c'(\rho) - 2c(\rho))} \rho^{-n}.$$

Bivariate example 1: Delannoy numbers

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- How to approximate a_{rs}?

Bivariate example 2: queueing network

- Consider

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- This corresponds to the recurrence (with appropriate boundary conditions)

$$9a_{rs} = 9a_{r-1,s} + 9a_{r,s-1} - 2a_{r-2,s} - 5a_{r-1,s-1} - 2a_{r,s-2}.$$

Naive attempts to generalize univariate methods fail badly

- Fix s and solve the problem for all r. In Delannoy example, need to compute
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- Even if we can understand the diagonal GF, its complexity grows with $r + s$. Also we can't derive asymptotics uniform in the slope, etc. Even the simplest examples (Delannoy, main diagonal gives $(1 - 6x + x^2)^{-1/2}$ are not so easy.

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- See our book: *Analytic Combinatorics in Several Variables*, Cambridge Studies in Advanced Mathematics 140, 2013.

Challenges

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- There are many areas in which to contribute. More (hu)manpower is needed!

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- Asymptotics of the monomer-dimer model on two-dimensional semi-infinite lattices. *Physical Review E*.

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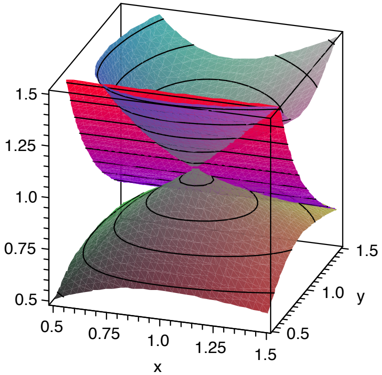
- We expect the singularities closest to the origin to play an important role. We use the Cauchy Integral Formula and adjust the contour near the boundary of the domain of convergence. This usually provides an exponential rate estimate.
- More detailed asymptotics requires detailed analysis of singularities of $\mathcal{V}$. Unlike the univariate case, rational functions can have nasty singularities. We have successfully analysed several important classes, but much work remains.

Example: $\mathcal{V}$ for "Arctic circle" dimer tiling model



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- For each $\mathbf{z}^* \in \mathrm{contrib}$, there is an asymptotic expansion $\mathcal{F}(\mathbf{z}^*)$ for $a_{\mathbf{r}}$, computable via derivatives of G and H.
- This yields

$$a_{\mathbf{r}} \sim \sum_{\mathbf{z}^* \in \mathrm{contrib}} {\mathbf{z}^*}^{-\mathbf{r}} \mathcal{F}(\mathbf{z}^*)$$

where $\mathcal{F}(\mathbf{z}^*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}^*$, and is uniform on compact subsets of directions provided the geometry does not change.

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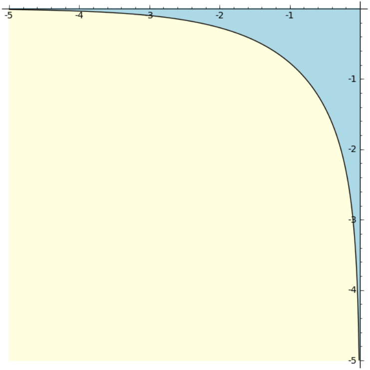
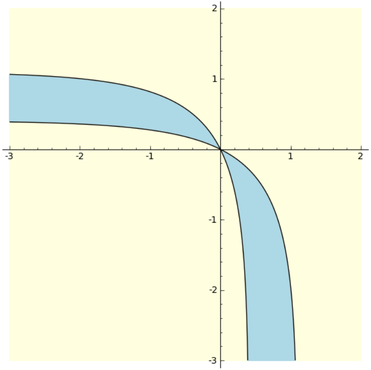
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \partial \log \mathrm{U}$ we denote by $\mathrm{K}(\mathbf{z}^*)$. If $\mathbf{z}^*$ is smooth, this is a single ray determined by $\mathrm{dir}(\mathbf{z}^*)$, the image of $\mathbf{z}^*$ under the logarithmic Gauss map.

Picture of $\log \mathrm{U}$ for Delannoy and queueing examples





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- We also have results for quadratic cone singularities.

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$$a_0 = G(\mathbf{z}^*)(\det J)^{-1}$$

where J is the Jacobian matrix $(\partial H_i/\partial z_j)$, other a_k are zero;

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where C depends on the derivatives to order 2 of H;

- (multiple point, $n = d$)

$$a_0 = G(\mathbf{z}^*)(\det J)^{-1}$$

where J is the Jacobian matrix $(\partial H_i / \partial z_j)$, other a_k are zero;

- (smooth point)

$$a_0 = \frac{G(\mathbf{z}^*)}{|| \operatorname{dir}(\mathbf{z}^*)|| \sqrt{(2\pi ||\mathbf{r}||)^d \, \mathcal{K}(\mathbf{z}^*)}}$$

where $\mathcal{K}$ is the Gaussian curvature of $\log \mathcal{V}$.

Summary: the generic combinatorial case

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- We can now use $\mathcal{F}(\mathbf{z}^*)$ to compute asymptotics in direction $\overline{\mathbf{r}}$. Provided the geometry does not change, the above expansion is uniform (over compact subsets) in $\overline{\mathbf{r}}$.

Examples: crit and $\mathrm{contrib}$

- (Delannoy) Here $\mathcal{V}$ is globally smooth and crit is given by $1 - x - y - xy = 0$ and $x(1+y)s = y(1+x)r$. There is a unique solution $(\frac{d-s}{r}, \frac{d-r}{s})$ (where $d := \sqrt{r^2 + s^2}$) for each r, s, where the outward normal to $\log \mathrm{U}$ is parallel to (r, s).

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- (queueing) Here $(1, 1)$ is a double point. If $1/2 < r/s < 2$, then asymptotics are controlled by $(1, 1)$. For other directions, a smooth minimal point on the relevant sheet of $\log \mathcal{V}$ controls asymptotics.

Generic case in dimension 2: explicit formula

- Suppose that $F = G/H$ has a simple pole at $P = (z^*, w^*)$ and $F(z, w)$ is otherwise analytic for $|z| \le |z^*|, |w| \le |w^*|$. Define

$$Q(z,w) = -A^2B - AB^2 - A^2z^2H_{zz} - B^2w^2H_{ww} + ABH_{zw}$$

where $A = wH_w, B = zH_z$, all computed at P. Then when $s \to \infty$ with $r/s = B/A$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*,w^*)}{\sqrt{2\pi}}\sqrt{\frac{-A}{sQ(z^*,w^*)}} + O(s^{-3/2})\right].$$

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  The apparent lack of symmetry is illusory, since $A/s = B/r$.
- This simplest case already covers Pascal, Catalan, Motzkin, Schröder, . . . triangles, generalized Dyck paths, ordered forests, sums of IID random variables, Lagrange inversion, . . . most published applications.

Example: Delannoy numbers

Recall

- Find rate of growth of $a_{r,s}$ as $d := \sqrt{r^2 + s^2} \to \infty$

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$$a_{100,100} \cong \frac{(1+\sqrt{2})^{201}}{10 \cdot 2^{5/4}\sqrt{\pi}} \quad \text{(accurate to within } 0.1\%.\text{)}$$

Example: queueing network

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- In the cone $1/2 < r/s < 2$, we have $a_{rs} \sim 3$ (note the error terms are exponentially small). Outside, the smooth formula holds.

Cauchy integral formula

- We have

$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$

where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.

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- This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.
- It is natural to try a saddle point/steepest descent approach.

The basic reductions

- We reduce the Cauchy integral by stratified Morse theory to iterated integrals over quasi-local cycles (up to exponentially smaller terms). This can all be done very concretely for smooth and multiple points.

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- The outer integral is now a Fourier-Laplace integral (after trigonometric substitution ($\mathbf{z} = \exp(i\boldsymbol{\theta})$).
- We derive asymptotics of the F-L integral by a version of the saddle point method (we needed to extend published results in some areas).

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta}) \, dV(\boldsymbol{\theta})$$

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- Difficulties in analysis: interplay between exponential and oscillatory decay, boundary terms, nonsmooth boundary of simplex.

Example: Delannoy numbers

- The relevant integral is

$$\int_D \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

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- Thus $f(0)=0$, and $f'(0)=0$ because (z^*,w^*) is a critical point for direction $\overline{(r,s)}$. This allows us to derive asymptotics of the right order.

Extensions

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- Dominant singularities that are not multiple points require more work. So far we have dealt with quadratic cone points reasonably well (Ch 11).
- If $a_{\mathbf{r}}$ is not nonnegative, then the elements of $\mathrm{contrib}$ need not be minimal points, and computing them is much harder. We have only looked at special examples (Ch 9.4).

Higher order asymptotics: Delannoy numbers $a_{3n,2n}$

n	1	2	4	8	16
exact	25	1289	$4.6733 \cdot 10^6$	$8.5276 \cdot 10^{13}$	$3.9780 \cdot 10^{28}$
1-term approx	26.263	1321.5	$4.7322 \cdot 10^6$	$8.5811 \cdot 10^{13}$	$3.9904 \cdot 10^{28}$
2-term approx	24.944	1288.4	$4.6728 \cdot 10^6$	$8.5273 \cdot 10^{13}$	$3.9780 \cdot 10^{28}$
1-term rel error	0.050525	0.025246	0.012597	0.0062895	0.0031420
2-term rel error	0.0022371	0.00050044	0.00011673	0.000028104	0.0000068844

Future work

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- Note that this approach would never occur to someone living only in the univariate world.
- There are some problems: the leading term for $b_{\mathbf{r}}$ vanishes (can deal with); not all $b_{\mathbf{r}}$ need be nonnegative, so dominant singularities are not necessarily minimal and may even lie at ∞ (don't know how to deal with in general). An interesting project and I am looking for collaborators.

Algebraic GF example via Safonov

- The Narayana numbers are generated by

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- This case was relatively easy because there was only one branch of the algebraic function passing through the origin.