A measure of the difficulty of manipulation of voting rules

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www.cs.auckland.ac.nz/˜mcw/blog/
(joint with Geoff Pritchard and Reyhaneh Reyhani)

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Our recent work

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RPW201x R. Reyhani, G. Pritchard and M. C. Wilson, A new measure of the difficulty of manipulation of voting rules, preprint 2009.

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- We break ties symmetrically: choose a tied winner uniformly at random. This is not an essential assumption but makes computation somewhat easier.
- We can describe the individual votes by a profile, an ordered list of the individual votes. There are $(m!)^n$ of these. For anonymous voting rules we need only the succinct input (voting situation) which lists numbers of voters of each preference. There are $\binom{n+m!-1}{n}$ of these.

Manipulation

- Let $X \subseteq V$. If $E_v \neq S_v$ for some $v \in X$, yet $E_v = S_v$ for all $v \in V \setminus X$, and each member of X prefers this to the sincere outcome, we say the profile is manipulable by X.

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- Note for experts: our random tie-breaking means G-S does not strictly apply, but a variant does. Our definition of manipulation does deal with ties.
- Key theme in most research literature: since manipulation is essentially unavoidable, how can we minimize its impact? In order to do this, we need to quantify manipulability.

Example: manipulation

- Consider the voting situation with with x bac and y cab voters, $x, y > 0$. Under antiplurality (veto rule) given by $w = (1, 1, 0)$, the sincere scores are $(x + y, x, y)$ and a wins.

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- Manipulability can be similarly described by more complicated systems of integer linear (in)equalities for most commonly used rules, including all scoring rules, Copeland's rule, etc.

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- "Lifted" random variables $\mathcal{I}, \mathcal{M}$ formed from these by sampling from the preference distribution. Note they can be defective.
- Distribution function of $\mathcal{M}$: probability that the situation is manipulable by k or fewer voters. More information than previous one, but still doesn't consider number of coalitions.

A new measure of manipulability (RPW201x)

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- Let Q be the random variable thus obtained, $\mathcal{Q}$ its lifting.
- It is easy to show that $\Pr(Q \leq k)$ equals the probability that a randomly chosen k-subset of V contains a manipulating coalition.
- Thus $\mathcal{Q}$ measures both the size and prevalence of manipulating coalitions. It is an average-case analogue of the best-case $\mathcal{M}$, and contains more information than the other measures.

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- Using the rule $w = (3, 2, 0)$ for the same situation, the sincere result is $(16, 18, 6)$ and b wins. Again $M = 2$, via two $cab \mapsto acb$ switches.
- Q can have any value between 2 and 8. Expected value of Q is 6.

Computation of Q

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- Differences from classical case: sampling without replacement, union not intersection.
- For rules amenable to linear system description, we are looking at a type of random walk and want the time to hit one of several polytopes.
- Can compute exactly in polynomial time in n for fixed m, but the obvious algorithm is $\Omega(n^5)$ even for $m = 3$.

Results for scoring rules

- Exact computation of distribution function of $\mathcal{M}$ (resp. $\mathcal{Q}$) for $m = 3$ up to $n = 150$ (resp. $n = 25$) for 6 rules, under 2 probability distributions [PrWi2007, RPW201x].

See the next talk for the last two points.

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- For IC (uniform profile distribution), an analytic description of asymptotic (in n) size of $\mathcal{M}$ for any fixed m [PrWi2009].
- We expect a similar result for $\mathcal{Q}$ (current work by PhD student Reyhaneh Reyhani).

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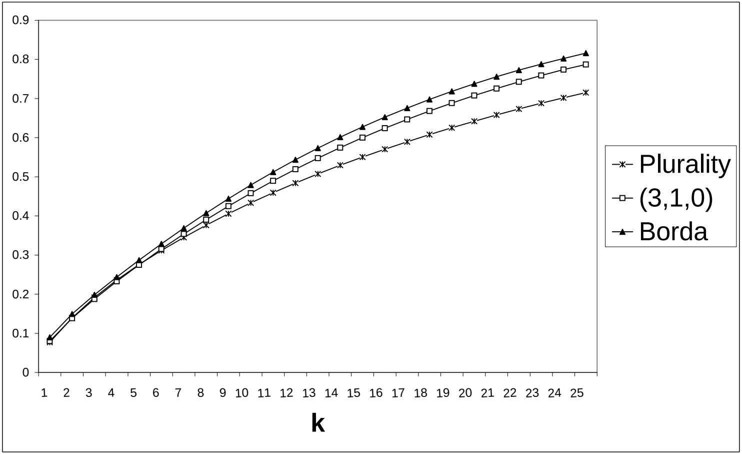
$\Pr(Q \leq k)$ for $n = 25$ under IC, $m = 3$



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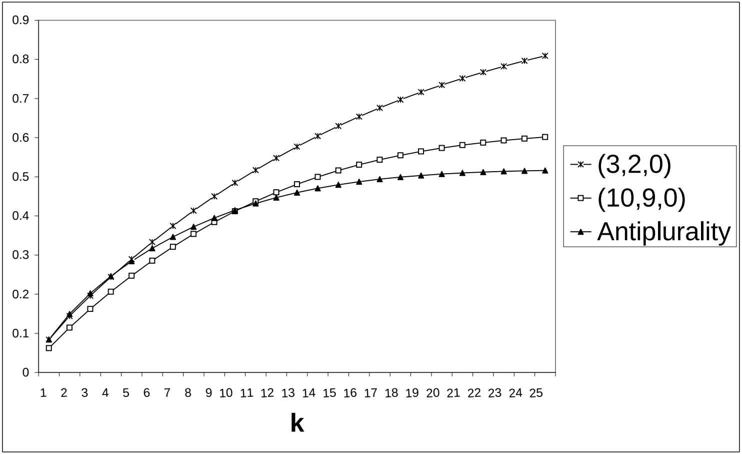

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- Slinko, Xia, Conitzer, Procaccia, Rosenschein, Zuckerman have discussed the phase transition under IC for $\Pr(\mathcal{Q} \leq k)$ as k increases past $\sqrt{n}$, for classes of rules including scoring rules. They often focus on weighted voting but the definition of manipulation is not always the same.

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- Walsh has discussed this phase transition in detail for specific rules, mostly using simulation and considering weighted voting.

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- Dowding and van Hees (British J. Politics, 2008) argue that encouraging strategic voting has many benefits for democracy. Buchanan and Yeo (Public Choice, 2006) argue that in fact all voting is strategic.