What is Analytic Combinatorics?

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- *Analytic Combinatorics* presumably means the use of mathematical analysis to study problems in combinatorics.
- Analysis has many branches: real, complex, functional, differential equations, measure theory, ... There are many possible ways to apply it to combinatorics!
- The most common usage of the term refers to the application of complex analysis to combinatorial enumeration (counting, discrete probability).
- The monograph *Analytic Combinatorics* (P. Flajolet and R. Sedgewick, 2009) has been very influential.

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Example

- Consider the Fibonacci numbers defined by the recurrence relation

$$a_n = \begin{cases} a_{n-1} + a_{n-2} & n \geq 2 \\ n & n \in \{0, 1\}. \end{cases}$$

- Abraham de Moivre introduced in 1730 the *generating function* method to obtain a solution to such recurrences.
- Write

$$F(x) = \sum_{n \geq 0} a_n x^n.$$

- We can think of this today as a transform of the sequence (a_n), analogous to a Fourier or Laplace transform.
- How de Moivre had the idea I don't know, but it has been amazingly fruitful. There are many later related constructions, such as exponential generating functions, Dirichlet series, ...

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- We obtain from the recurrence

$$\begin{aligned} F(x) &= \sum_{n\geq 0} a_n x^n \\ &= 0 + 1\cdot x + \sum_{n\geq 2} a_n x^n \\ &= x + \sum_{n\geq 2} (a_{n-1} + a_{n-2})\, x^n \\ &= x + \sum_{n\geq 1} a_n x^{n+1} + \sum_{n\geq 0} a_n x^{n+2} \\ &= x + x(F(x) - 0) + x^2 F(x). \end{aligned}$$

- Thus we have an explicit rational function

$$F(x) = \frac{x}{1 - x - x^2}.$$

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Extracting coefficients: lookup table

- We can now exploit the rational function to obtain exact and asymptotic formulae for the coefficients.
- Partial fraction decomposition yields

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\theta x} - \frac{1}{1+\theta^{-1}x}\right)$$

where

$$\theta = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is the reciprocal of the positive root of the denominator.

- Thus by geometric series expansion

$$a_n = \frac{1}{\sqrt{5}}\left(\theta^n - (-\theta)^{-n}\right) \sim \frac{1}{\sqrt{5}}\theta^n.$$

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Extracting coefficients: inverting the transform

- A more general method uses Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}} F(z) z^{-n} \, \frac{dz}{z}$$

for some small circle $\mathcal{C}_\varepsilon$ of radius ε enclosing the origin.

- We can expand it just beyond $z = \theta^{-1}$, picking up a residue:

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}_{\theta^{-1}+\varepsilon}} F(z) z^{-n} \, \frac{dz}{z} - \operatorname{Res}(F(z) z^{-n-1}; \theta^{-1}).$$

- The residue at the simple pole is easily calculated as

$$(\theta^{-1})^{-n} \operatorname{Res}(zF(z); \theta^{-1}) = \theta^n \lim_{z \to \theta^{-1}} (z - \theta^{-1}) z F(z) = -\theta^n \frac{1}{\theta + \theta^{-1}}$$

and the integral is exponentially smaller because it is bounded by $K(\theta^{-1} + \varepsilon)^{-n}$. Thus we eventually get the same asymptotic result.

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Example

Another famous example of this technique is the sequence of Catalan numbers with recurrence

$$c_n = \begin{cases} \sum_{j=0}^{n-1} c_j c_{n-1-j} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

and corresponding generating function

$$\frac{1 - \sqrt{1-4x}}{2x}.$$

They count enormously many interesting things; see book by Stanley. In this case

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

but such nice formulae are rare.

Various data structures for sequences

- We can describe a sequence of interest by a defining recurrence, generating function, exact formula, or asymptotic formula. The recurrence (with initial conditions) and the GF each uniquely determine the sequence, and each other.
- Standard operations on sequences correspond directly to standard operations on GFs (e.g sum to sum, convolution to product, sequence to quasi-inverse), so we have a dictionary translating between them.
- Each recurrence translates via this dictionary to functional equation for the GF.
- Thus if we can specify the combinatorial class we are studying by building it up from simpler ones via standard operations, we can compute an explicit formula for the GF.
- Experience shows that the GF is the best all-round data structure.

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The key principles of analytic combinatorics

- location of dominant points of the singular variety $\mathcal{V}$ of the GF determines exponential growth rate of coefficients;
- type of singularity determines polynomial corrections.

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- type of singularity determines polynomial corrections.

Hierarchy of GFs

- Rational functions correspond to linear recurrences with constant coefficients. Sample applications: patterns in strings/finite automata, paths in graphs/transfer matrix method.
- Algebraic functions arise from context-free languages (Chomsky-Schützenberger), for example many problems involving trees. There is no nice description of the recurrences they correspond to.
- D-finite (or holonomic) functions satisfy a linear ODE with polynomial coefficients. They correspond to linear recurrences with polynomial coefficients.
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Asymptotics from univariate GFs

- A rational GF generically has coefficients that look like αC^n where $1/C$ is the smallest modulus root of the denominator.
- Catalan numbers have an algebraic irrational GF, the asymptotics look like $4^n/\sqrt{\pi n^3}$. This is typical.
- D-finite sequences typically look asymptotically like algebraic ones, with possibly a power of $\log n$ as an extra factor.
- Periodicity in sequences can be handled easily, as can exponentially smaller terms.
- The univariate case can be algorithmically handled for algebraic GFs, but some issues remain in the D-finite case.

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Multivariate case

- Deriving multivariate GFs is often not much harder than in the univariate case.
- However analysing them is much harder, even for rational functions:
  - Algebra: partial fraction decomposition does not apply in general.
  - Geometry: the singular variety $\mathcal{V}$ is more complicated.
    - it does not consist of isolated points, and may self-intersect;
    - real dimension of contour is d, that of $\mathcal{V}$ is $2d-2$, so less room to avoid each other;
    - Topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated.
  - Analysis: the (Leray) residue formula is much harder to use.
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Example (Delannoy walks)

- We count walks on the lattice $\mathbb{Z}^2$ starting at $(0,0)$ and ending at (r,s), with each step chosen from $\{\uparrow, \rightarrow, \nearrow\}$.
- The recurrence is

$$a_{rs} = \begin{cases} a_{r-1,s-1} + a_{r-1,s} + a_{r,s-1} & \text{if } r > 1, s > 1 \\ 1 & (r,s) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

  and GF is $(1 - x - y - xy)^{-1}$.
- How to compute asymptotics for $a_{r,s}$ for large $r + s$?

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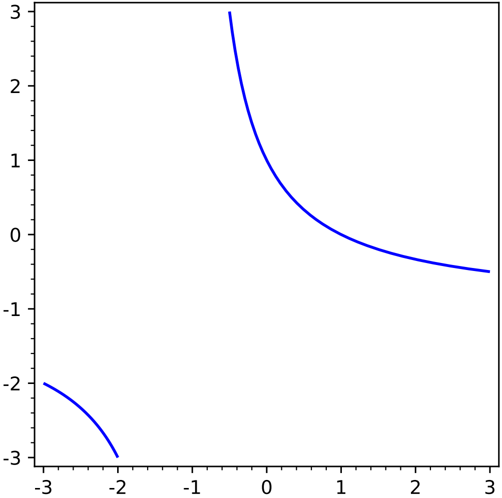
$$a_{rs} = \begin{cases} a_{r-1,s-1} + a_{r-1,s} + a_{r,s-1} & \text{if } r > 1, s > 1 \\ 1 & (r,s) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

  and GF is $(1 - x - y - xy)^{-1}$.
- How to compute asymptotics for $a_{r,s}$ for large $r + s$?

Example (Delannoy walks: singular variety)

The complex curve given by $1 - x - y - xy = 0$ (real points shown).



Multivariate problems include univariate ones

- The diagonal of the d-variate GF

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$$

is the univariate GF

$$\Delta F(z) = \sum_{n} a_{n,n,\ldots,n} z^{n}.$$

- If F is rational and $d = 2$, then ΔF is algebraic; conversely every algebraic function arises in this way.
- If F is rational then ΔF is D-finite. Most D-finite functions in applications arise in this way.

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Sample of interesting multivariate problems

- Random walk on a lattice: find probability of return to the origin.
- Quantum random walk in d dimensions: determine the feasible region.
- Tiling models from statistical physics: determine the frozen region.
- Partition functions in queueing networks: asymptotic approximation to values.
- Probability limit laws (e.g. the number of occurrences of a fixed substring in a random string is asymptotically normally distributed).

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General procedure for coefficient asymptotics

- Express $a_{\mathbf{r}}$ using the Cauchy Integral Formula.
- Choose direction of asymptotics.
- Use residue theory to express $a_{\mathbf{r}}$ in that direction in terms of integrals over small cycles around contributing points.
- Asymptotically approximate by the dominant integrals.
- Evaluate the dominant integrals somehow!

The third and fifth steps are the hardest. The fifth step depends greatly on the local geometry of the singularity. In the univariate case, the second and third steps are often rather trivial, and the fifth is completely unnecessary.

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Outline of (generic) results achieved so far in ACSV project

- Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\bar{\mathbf{r}})$, of critical points. The set $\mathrm{crit}(\bar{\mathbf{r}})$ is computable via polynomial algebra.
- For computing asymptotics in direction $\bar{\mathbf{r}}$, we may restrict to a dominant point $\mathbf{z}_*(\bar{\mathbf{r}})$ lying in the positive orthant.
- There is an asymptotic series $\mathcal{A}(\mathbf{z}_*)$ for $a_{\mathbf{r}}$, depending on the type of geometry of $\mathcal{V}$ near $\mathbf{z}_*$, and each term is computable from finitely many derivatives of G and H at $\mathbf{z}_*$.
- This yields

$$a_{\mathbf{r}} \sim \mathcal{A}(\mathbf{z}_*)$$

where the expansion is uniform on compact cones of directions, provided the geometry does not change.

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Asymptotic formula in simplest case

Theorem

Suppose that $F = G/H$ *has a strictly minimal simple pole at* $\mathbf{p} = (z^*, w^*)$. *If* $Q(\mathbf{p}) \neq 0$, *then when* $s \to \infty$ *with* $(rwH_w - szH_z)_{|\mathbf{p}} = 0$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{2\pi}}\sqrt{\frac{-wH_w(\mathbf{p})}{sQ(\mathbf{p})}} + O(s^{-3/2})\right].$$

- The apparent lack of symmetry is illusory, since $wH_w/s = zH_z/r$ at $\mathbf{p}$.
- Here $\mathbf{p}$ is given, which specifies the only direction in which we can say anything useful.
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Next simplest case

Theorem

Suppose that $F = G/H$ has a strictly minimal pole at $\mathbf{p} = (z_*, w_*)$, which is a double point of $\mathcal{V}$ such that $G(\mathbf{p}) \neq 0$. Then there is a cone $\mathrm{K}(\mathbf{p})$ such that as $s \to \infty$ for r/s in $\mathrm{K}(\mathbf{p})$,

$$a_{rs} \sim (z_*)^{-r}(w_*)^{-s}\left[\frac{G(\mathbf{p})}{\sqrt{(z_* w_*)^2\,\mathrm{Q}(\mathbf{p})}} + O(e^{-c(r+s)})\right]$$

where Q is the Hessian of H.

- Note that
  - the expansion holds uniformly over compact subcones of $\mathrm{K}(\mathbf{p})$;
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Extensions that we know how to do

- Arbitrary dimension, smooth/multiple point geometry of singularities
- Can get nastier local geometry of singularities (e.g. lattice tiling models)
- Can get continuum of contributing points (e.g. quantum random walks, vector partition functions)
- Need higher order asymptotics (e.g. cancellation in computing variance)

Extensions that we don't yet know how to do

- Algebraic GFs (some ideas on representation as diagonals of rational functions)
- Limiting directions (how do asymptotics patch together on boundaries of cones?)
- Noncombinatorial problems (finding contributing points is harder; need Morse theory, probably)
- Complicated local geometry of singularities (arises in some tiling models)

References

- ACSV project website: `acsvproject.org`.
- R. Pemantle & M.C. Wilson. *Analytic Combinatorics in Several Variables*. Cambridge University Press, 2013. (2nd edition being worked on now with S. Melczer)
- S. Melczer. *An Invitation to Analytic Combinatorics: From One to Several Variables*. Springer, to appear 2021.

- Contact me if you are interested in learning more!