New algorithms for one-sided matching

Mark C. Wilson

UMass Amherst
Department of Mathematics and Statistics
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- The same basic problem (with some variations) occurs in matching interns to hospitals, military staff to bases, children to public schools, volunteer teachers to schools, ...

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- We stick to the deterministic case today - random allocations are also very interesting but lead to many subtleties.

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- Boston school choice - allocate as many agents to their first choice as possible, then as many of those remaining to their second, etc, breaking ties according to a fixed order of agents.

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- Boston gives $1 : a, 2 : c, 3 : b$.

Properties of solutions

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- Strategyproofness - no incentive for agents to lie about preferences.
- Fairness - no agent envies another's item.
- Welfare - a combination of efficiency and fairness, measuring overall happiness of agents.

Properties of algorithms

Algorithm	Fast	Efficient	Strategyproof	Envy-free
SD	✓	✓	✓	✗
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- No algorithm can be envy-free in every situation.
- Thus we need to consider tradeoffs, and maybe there are new algorithms that make a good compromise.

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- My main idea: good algorithms probably exist in the "interior of algorithm space", and we should look for them.
- We should search for algorithms that behave "well" on "most" inputs.

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- Termination occurs when there are no unmarked items left.

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- In round 3, 3 takes b, 1 takes a, 2 takes b, 3 takes a, 1 takes c.

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- In round 1, 1 takes a.
- In round 2, 2 takes a, then 1 takes b.
- In round 3, 3 takes b, 1 takes a, 2 takes b, 3 takes a, 1 takes c.
- Final allocation: $1 : c, 2 : b, 3 : a$.

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- At first sight, YS doesn't seem very competitive. However . . .

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- The resulting directed graph must have a cycle (why?). We can reallocate along a cycle to make those agents happy.
- Remove the reallocated items and agents, and continue (starting with 2nd preferences) until no more cycles exist.
- This runs in polynomial time, and yields an efficient allocation.

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YS	✓	✗	✗	✗
YS+TTC	✓	✓	✗	✗

- YS followed by TTC is now efficient. Experiments show it outperforms SD (and usually Boston) in welfare and envy. It seems hard to manipulate strategically and feels “fairer” because if an item is taken from us, we can steal it back later.

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- Each agent may propose at most once to each item (so rejections are permanent), which guarantees termination.

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- To get more interesting algorithms, we can modify the items' preferences dynamically based on the proposals received.
- Main ideas: accept first (prefer agents in order they have proposed) and accept last (prefer them in reverse order). There may be other good ideas.
- Yankee Swap uses accept-last, since the current agent trying to steal an item is always accepted. SD and Boston use accept-first.

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- Serial Dictatorship and Boston have permanent memory, Yankee Swap has temporary memory.

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- In our dynamic preference setup it does. There are two obvious choices:
  - Queue-based: first in, first out;
  - Stack-based: last in, first out.
- Serial Dictatorship uses stack, Boston uses queue, Yankee Swap uses stack.

Old algorithms in this framework

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- We can also append TTC to any of the algorithms.
- We can show that all the Accept-First ones are efficient, and none of the Accept-Last ones are.
- This yields $2^3 + 4 = 12$ algorithms, so there are 9 more to discuss.

Behavior of algorithms on standard profile: 3 agents $a \succ b \succ c \succ d$, 1 agent $b \succ a \succ c \succ d$

Algorithm	Output matching	Number of proposals
PFS	1:a, 2:b, 3:c, 4:d	10
PFQ	1:a, 2:c, 3:d, 4:b	9
PLS	1:d, 2:c, 3:a, 4:b	9
PLQ	1:d, 2:c, 3:b, 4:a	10
TFS	1:d, 2:a, 3:c, 4:b	18
TFQ	1:a, 2:b, 3:d, 4:c	33
TLS	1:b, 2:a, 3:d, 4:c	18
TLQ	1:a, 2:b, 3:d, 4:c	21

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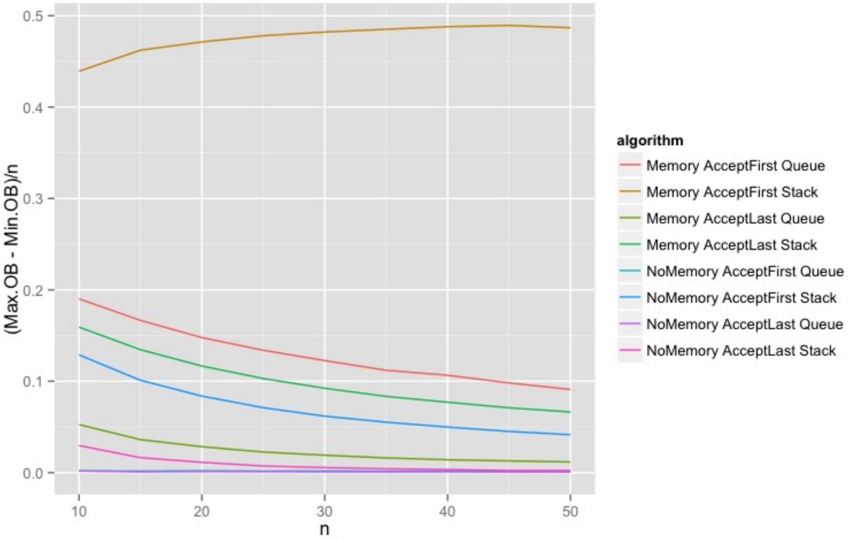
- We call an algorithm order-fair if for all i, j the expected rank of the item allocated to the ith agent is the same as for the jth agent, as we average over all preference profiles.
- SD is very far from order-fair, since the last agent is much worse off than the first one. Boston is better, but still heavily biased toward early agents.

Normalized order bias of our 8 basic algorithms



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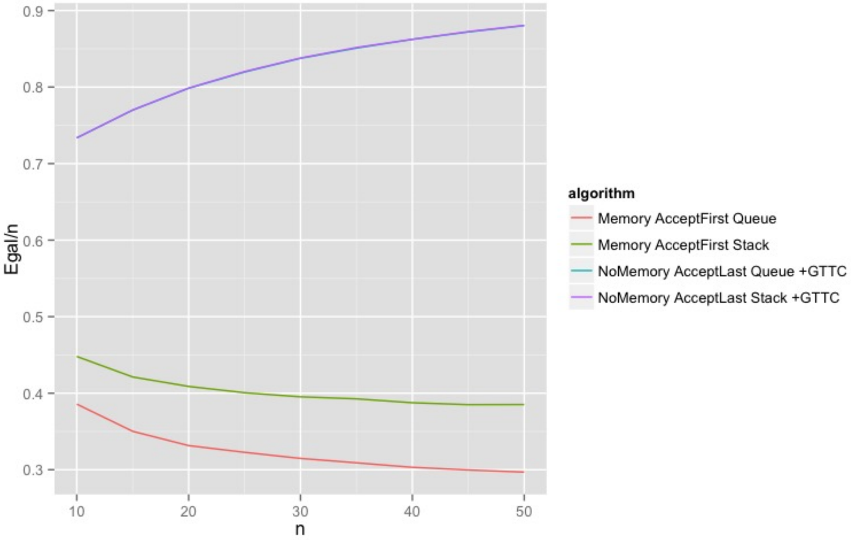
- The temporary-memory equivalent of SD proceeds like SD in each round. It favours the last agent over all the others, and may be appropriate for situations in which a single agent should be privileged.
- Interestingly, the queue-based algorithms are much fairer than the stack-based ones, and the queue analogue of Yankee Swap has amazingly small bias.

Egalitarian welfare of selected algorithms



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- We can create random allocation algorithms by simply randomizing the agent order, making it easier to avoid envy.
- There is still much that is unexplored. Who knows what other algorithms are out there?