# Some topics in network science

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  - Construction of a network from real data, some basic network science tools.
  - Social learning and "wisdom of crowds".
  - Structural balance in signed networks.

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- A relatively new example is the corpus of legislative documents (acts, regulations, case law). My PhD student Neda Sakhaee and I looked at New Zealand Acts of Parliament (in progress).
- Basic questions: what is the network structure? how does it evolve? which are the "most important/influential" documents? do they cluster?

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- Luckily there is a master title list of laws, and the NZ government makes *current* laws available in XML format.

NEW ZEALAND.

TRICESIMO NONO

# VICTORIÆ REGINÆ.

No. LXXIV.

ANALYSIS.



AN ACT to provide for the Collection by means of Stamps of Fees payable in the various Departments of the Public Service. Title.

[21st *October*, 1875.]

BE IT ENACTED by the General Assembly of New Zealand in Parliament assembled, and by the authority of the same, as follows:—

**1.** The Short Title of this Act shall be "The Stamp Fee Act, 1875." Short Title.

```
10.
Part I. of "Stamp Act, 1875," to be read al part of this Act. '


AN ACT to provide for the Collection by means of Titlo. Stamps of Fees payable in the
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1.
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2.
"The Supreme Court and Registration Offices Fees Act, 1866," Repeal. is hereby repealed.

3.
The Governor in Council may, by notice published in the New Governor may fix Zealand
Gazette, direct that after the time specified in such notice Atimte,f∞tor bringit~g


d t' f fi l' .c th t' b . C lD opera Ion
all     or any 0 f the u les ees nes or pena tIes J.or e Ime emg in any Department.
payable in money in any Public Department or office connected with
the public service, or to the officers thereof, shall be collected by
means of stamps; and after the time so specified, the duties fees
fines or penalties therein mentioned shall be received by stamps de≠
noting the sums payable and not in money.
4. The Governor in Council may make alter or repeal Regula-Governor may make Hons not
contrary to this Act for the due administration thereof. Regulations.
Sl~pplelnent to the New Zealand Gazette, No. 59, oftlte 2ht OcfollFr, I8i5.

39∞ VICTORIlE. No. 74.

Stamp Fee Act.
Stamps to be im-5. All or any stamps to be used under this Act shall be impressed
preGssed or addh.esivte or adhesive as the Governor from time to time directs.
as overnor Iree B. 6    . d' h" bl'
Stamp to be affixed to ï When any sum comprIse III any suc notICe IS paya e III
orimpress~duponthe respect of a document the stamps denoting such sum shall be affixed
document ill respect    ï
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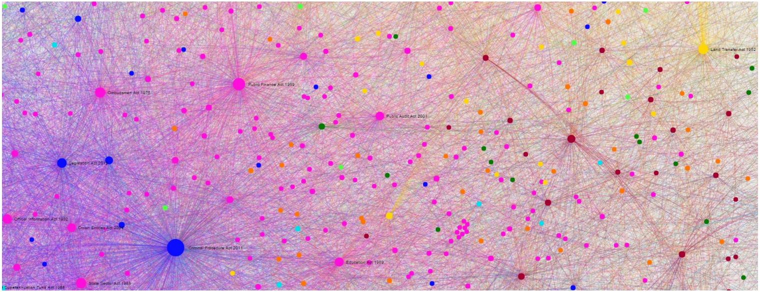
# Part of the network – color:community, node size:centrality




There are 10000s of nodes,100000s of edges.

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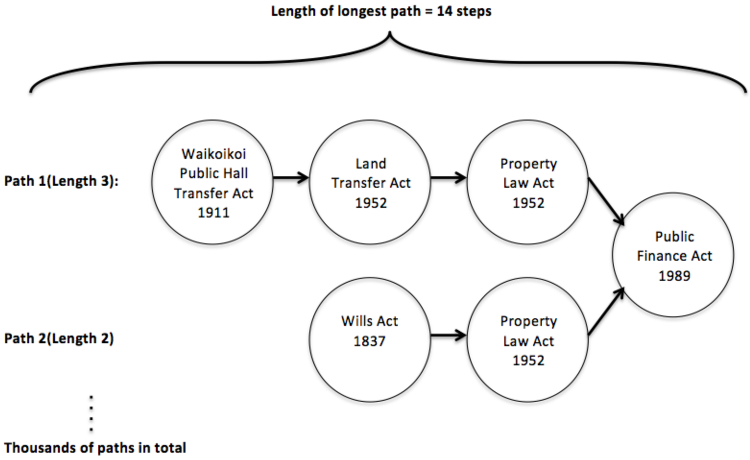
  - $C_K$, Katz centrality (weighted sum of paths to the node, weights decrease exponentially by length) is given by $C_K = ((I - \alpha A^T)^{-1} - I)\mathbf{1}$ for small $\alpha$.
- We found broad agreement between measures on the most important nodes and on the least important, *without any analysis of the content of documents*. This has been corroborated by expert opinion.


Length of longest path = 14 steps
Path 1(Length 3):
Waikoikoi Public Hall Transfer Act 1911
Land Transfer Act 1952
Property Law Act 1952
Public Finance Act 1989
Path 2(Length 2)
Wills Act 1837
Property Law Act 1952
Thousands of paths in total

| Act | Rank | $C_K$ |
|---|---|---|
| Public Finance Act 1989 | 1 | 10.37 |
| Criminal Procedure Act 2011 | 2 | 9.65 |
| Summary Proceedings Act 1957 | 3 | 9.28 |
| State Sector Act 1988 | 4 | 8.85 |
| District Courts Act 1947 | 5 | 7.96 |
| Crimes Act 1961 | 6 | 7.47 |
| Companies Act 1993 | 7 | 7.43 |
| Local Government Act 1974 | 8 | 7.4 |
| Judicature Act 1908 | 9 | 7.1 |
| Privacy Act 1993 | 10 | 6.79 |
| Resource Management Act 1991 | 11 | 6.71 |
| Official Information Act 1982 | 12 | 6.58 |

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- Detect communities (unusually dense subgraphs) - challenging because network is directed.
- Comparative studies with other jurisdictions — how much can be read off just from the citation network?
- Other layers (regulations, case law) of the network.

# References

- N. Sakhaee, M.C. Wilson, G.Zakeri. *Structural Analysis of Legislation Networks*. Proceedings JURIX 2016.

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# Basic setup for discrete time diffusion models

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- At each discrete time step, update a node by a fixed function of the colors of its neighboring nodes.
- We study dynamics of the profile of node states. Analytic results are hard for all but the easiest models.

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- (homophily) Describe the effect on the process of assuming that nodes of same color are more likely to be connected.
- (cascades) When do arbitrary changes to some nodes propagate to a large fraction of the network?

# Some discrete time belief change models

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- The last two can easily oscillate depending on topology and initial coloring. Note that these differ markedly from standard probabilistic contagion models for disease.

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- Although this is a very restricted environment, I think it has some relevance to online social networks and political discussion. It is relevant to multiagent intelligent systems.

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  - The name of the character played by Paul Walker in "The Fast and the Furious" is "Dominic".

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- Subjects answer "don't know" much less often than expected.
- Subjects admitting "don't know" at first iteration learned much more often than those who answer wrongly.
- There is much more apparent social influence than we expected on logical questions.
- Difficult questions may lead to good social learning, but "tricky" questions (where subjects don't know they don't know) lead to really bad social learning.
- Promising new model to study: switching probability from "yes" to "no" is proportional to $(p_B^2 - p_W^2)$.

# References


- P. Girard, V. Pavlov, M.C. Wilson. *Networked crowds answer tricky questions poorly*. Preprint 2016.

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- P. Girard, V. Pavlov, M.C. Wilson. *Belief diffusion in social networks*. Preprint 2015.
- Could use help on methodology: how do we falsify a model? how do we get enough data to test a model? what techniques of statistical inference are appropriate?

# Balance in signed networks

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- $G$ is balanced iff all its cycles are balanced.
- Real-world networks are rarely balanced.

# Properties equivalent to balance

- $V = V_0 \cup V_1$ such that $(x, y) \in E_-$ implies $x \in V_i, y \in V_{1-i}$ (polarization).

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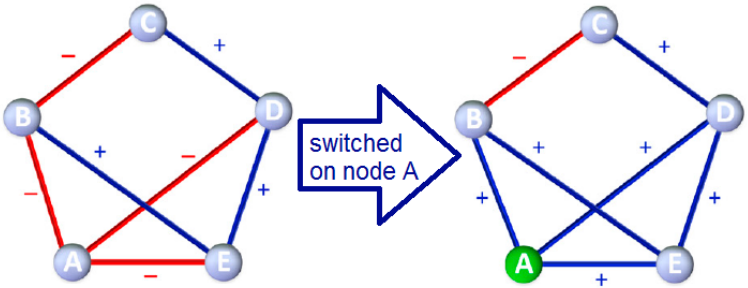
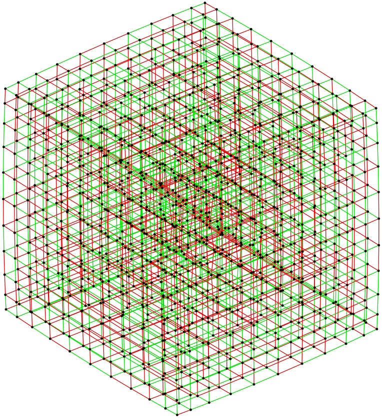
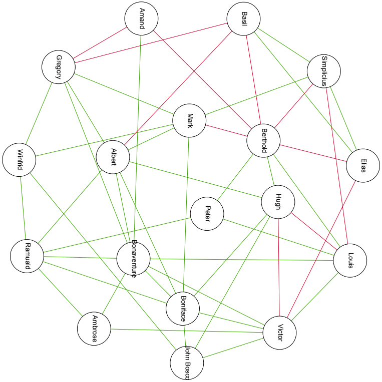
- $(V, E_-)$ is bipartite.
- $G$ is switching equivalent to a graph with all positive edges.
- The smallest eigenvalue of the Laplacian $D - A$ (where $D$ is the diagonal matrix of degrees) of $G$ is $0$.

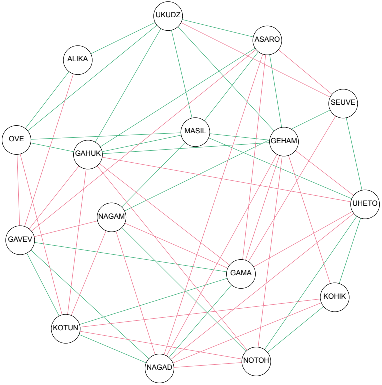
# Switching



# Examples of real world signed networks






UKUDZ
ASARO
ALIKA
SEUVE
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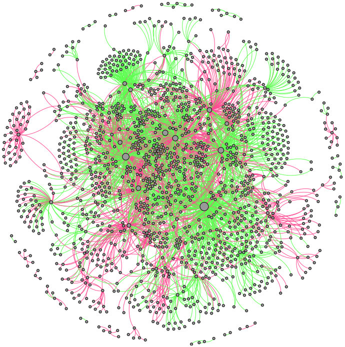

# Middle East signed network

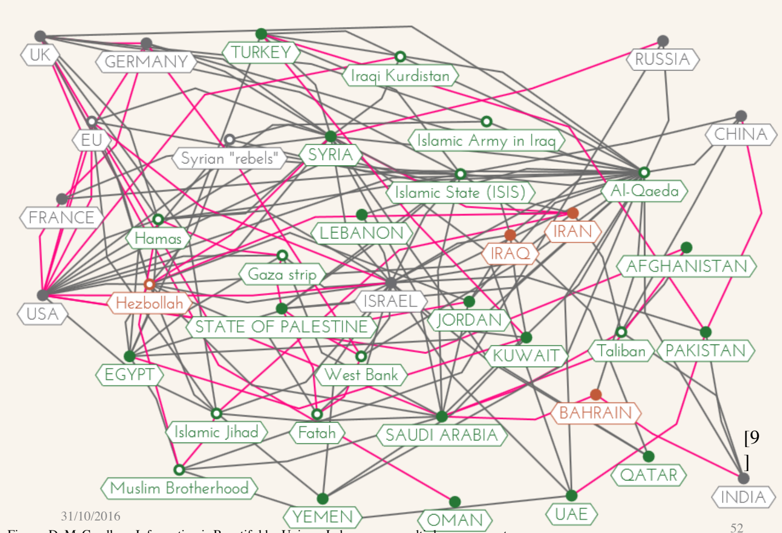


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Figure: D. McCandless, Information is Beautiful by Univers Labs, source: multiple news reports

# How to measure partial balance?

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- Real networks are not usually balanced, but there are theories that they become more balanced over time.
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- We reviewed several measures, introduced axioms and desirable properties, and studied them thoroughly on synthetic and real data.
- One of the best performing measures: the frustration index, a normalization of the minimum number of edges we must flip/delete in order to achieve perfect balance.
- We show that well known and commonly used measures such as the fraction of balanced cycles have serious drawbacks.

# Axioms

- A1 $0 \leq \mu(G) \leq 1$.
- A2 $\mu(G) = 1$ if and only if $G$ is balanced.
- A3 If $\mu(G) \leq \mu(H)$, then $\mu(G) \leq \mu(G \oplus H) \leq \mu(H)$.
- A4 $\mu(G^{g(X)}) = \mu(G)$.
- B1 If $\mu(G) \neq 1$, then $\mu(G \oplus C_3^+) > \mu(G)$.
- B2 If $\mu(G) \neq 0$, then $\mu(G \oplus C_3^-) < \mu(G)$
- B3 If $e \in E^*$, then $\mu(G \ominus e) \geq \mu(G)$.
- B4 If $\mu(G) \neq 0$ and $\mu(G \ominus E^* \oplus e) \neq 1$, then $\mu(G \oplus e) \leq \mu(G)$.

# Axiomatic behavior of measures

| | $D(G)$ | $C(G)$ | $W(G)$ | $D_k(G)$ | $A(G)$ | $F(G)$ |
|---|---|---|---|---|---|---|
| A1 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| A2 | ✓ | ✓ | ✓ | ✗ | ✗ | ✓ |
| A3 | ✓ | ✓ | ✓ | ✓ | ✗ | ✓ |
| A4 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| B1 | ✓ | ✓ | ✓ | ✗ | ✓ | ✓ |
| B2 | ✓ | ✓ | ✗ | ✗ | ✗ | ✗ |
| B3 | ✗ | ✗ | ✗ | ✗ | ✗ | ✓ |
| B4 | ✗ | ✗ | ✗ | ✗ | ✗ | ✓ |

# Balance in minimally and maximally unbalanced $K_n$

| $\mu(G)$ | $\mu(G_{\min})$ | $\mu(G_{\max})$ |
|---|---|---|
| $D(G)$ | $\sim 1 - 2/n$ | $\sim \frac{1}{2} + (-1)^n e^{-2}$ |
| $C(G)$ | $\sim 1 - 1/n$ | $\sim \frac{1}{2} - \frac{3n \log n}{2^n}$ |
| $D_k(G)$ | $1 - 2k/n(n-1)$ | $0, 1$ |
| $W(G)$ | $\sim 1 - 2/n$ | $\sim \frac{1+e^{2-2n}}{2}$ |
| $A(G)$ | $\sim 1 - 4/n^2$ | $0$ |
| $F(G)$ | $1 - 4/n(n-1)$ | $\frac{1}{n}, \frac{1}{n-1}$ |

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  - using data reduction (preprocessing)
  - reformulating our original nonlinear model as a linear one
  - using the structure of the problem to create nonobvious constraints
  - using IP techniques ("lazy cuts")

# XOR model

$$
\begin{aligned}
\min_{x_i : i \in V, f_{ij} : (i,j) \in E} Z = & \sum_{(i,j) \in E} f_{ij} \\
\text{s.t.} \quad & f_{ij} \geq x_i - x_j \quad \forall (i,j) \in E^+ \\
& f_{ij} \geq x_j - x_i \quad \forall (i,j) \in E^+ \\
& f_{ij} \geq x_i + x_j - 1 \quad \forall (i,j) \in E^- \\
& f_{ij} \geq 1 - x_i - x_j \quad \forall (i,j) \in E^- \\
& x_i \in \{0,1\} \quad \forall i \in V \\
& f_{ij} \in \{0,1\} \quad \forall (i,j) \in E
\end{aligned} \tag{1}
$$

# Some additional constraints

Feasible:

$$f_{ij} + f_{ik} + f_{jk} \geq 1 \quad \forall (i,j,k) \in T^-$$

Optimal:

$$\sum_{j:(i,j)\in E \text{ or } (j,i)\in E} f_{ij} \leq (d_i/2) \quad \forall i \in V$$

# Results

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- The current implementations are at least 10 times faster than the original and allow computation in networks with thousands of nodes and edges.
- They are the best we know of by quite some distance (several orders of magnitude faster on test problems).
- We show that many previously computed results are incorrect.

# Sample results

| | Graph | D2007 | H2010 | I2010 | XOR |
|---|---|---|---|---|---|
| Quality | EGFR | [196, 219] | 210 | [186, 193] | 193 |
| | macrophage | [218,383] | 374 | [302, 332] | 332 |
| | yeast | [0, 43] | 41 | 41 | 41 |
| | E.coli | [0, 385] | fail | [365, 371] | 371 |
| Time | EGFR | 420 s | 6480 s | >60 s | 0.28 s |
| | macrophage | 2640 s | 60 s | >60 s | 0.56 s |
| | yeast | 4620 s | 60 s | >60 s | 0.13 s |
| | E.coli | - | fail | >60 s | 2.21 s |

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- However certain biological networks are much less balanced than expected.
- International alliances network seems to become (slowly) more balanced over time.

# References

- S. Aref, M.C. Wilson. *Measuring Partial Balance in Signed Networks*. Accepted Journal of Complex Networks 2017.

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