Asymptotics via multivariate generating functions

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- How to extract information about a_{rs} as $r+s \to \infty$?

Standing assumptions

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.

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- Assume $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ where G, H are analytic (e.g. polynomials). The singular variety $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$ consists of poles.

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  - the definition of "dominant" is a little different;
  - the local geometry of $\mathcal{V}$ can be much nastier;
  - residue computations are harder and residues must still be integrated.

Example (Univariate pole: derangements)

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole, at $z = 1$. Using a circle of radius $1 - \varepsilon$ yields, by Cauchy's theorem

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- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.

Example (Essential singularity: saddle point method)

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- Now minimize the height function $h_n(R)$ over R. In this example, $R = n$ is the minimizer.
- The integral over C_n has most mass near $z = n$, so that

$$a_n = \frac{F(n)}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) \frac{F(ne^{i\theta})}{F(n)}\, d\theta$$

$$\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp\left(-in\theta + \log F(ne^{i\theta}) - \log F(n)\right)\, d\theta.$$

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- This gives, with $b_n = 2\pi n^n e^{-n} a_n$, Laplace's approximation:

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- This recaptures Stirling's approximation, since $n! = 1/a_n$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

Multivariate asymptotics — some quotations

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- We aimed to improve the multivariate situation.

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- We relax several of these assumptions later.

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  that is uniform on compact subsets of directions, provided the geometry does not change.
- The set $\mathrm{crit}(\overline{\mathbf{r}})$ is computable via symbolic algebra.
- To determine the dominant point requires a little more work.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) \, \mathbf{dz}$$
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- To derive asymptotics, it is natural to try a saddle point/steepest descent approach.

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where C_i is a quasi-local cycle near some critical point ${\mathbf{z}_*}^{(i)}$.

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- The critical points are those where the restriction of h to a stratum has derivative zero.
- Key problem: find the highest critical points with nonzero n_i. These are the dominant ones.

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- We can (with some effort) convert quantities in our formula back to the original data.

Fourier-Laplace integrals

- We ultimately reduce to asymptotics for large λ of

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  - f decays exponentially on ∂D, or the amplitude A vanishes there;

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  - f has an isolated quadratically nondegenerate stationary point.
- Many of our applications to generating function asymptotics do not fit into this framework, and we needed to extend what is known — for this analyticity was very useful.

Low-dimensional examples of F-L integrals

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- Multiple point with $n = 2, d = 1$ gives integral like
$$\int_{-1}^{1}\int_{0}^{1}\int_{-x}^{x} e^{-\lambda(z^2+2izy)}\,dy\,dx\,dz.$$
Simplex corners now intrude, continuum of critical points.

Logarithmic domain

- Let U be the domain of convergence of the power series $F(\mathbf{z})$. We write $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the logarithmic domain of convergence. This is known to be convex.

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- For each $\overline{\mathbf{r}}$ we can find $\mathbf{z}_*(\overline{\mathbf{r}}) = \exp(\mathbf{x}^*)$, on the boundary of $\mathcal{V}$ and in the positive orthant of $\mathbb{R}^d$, that controls asymptotics in direction $\overline{\mathbf{r}}$.

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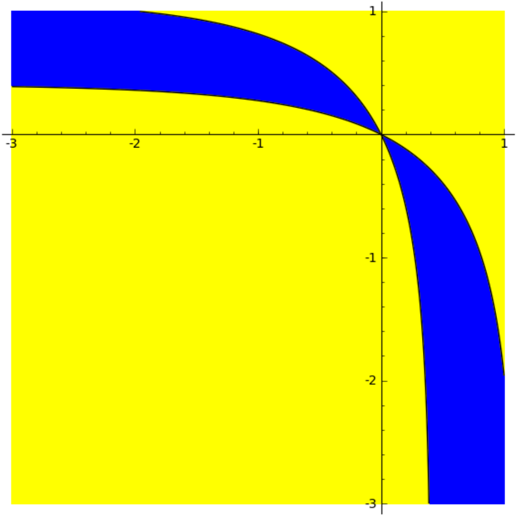
- We denote by $\mathrm{K}(\mathbf{z}_*)$ the cone spanned by normals to supporting hyperplanes at $\mathbf{x}^*$. If $\mathbf{z}_*$ is smooth, this is a single ray determined by the image of $\mathbf{z}_*$ under the logarithmic Gauss map $\nabla_{\log} H$.

$\log U$ for queueing example



Generic case — smooth point formula for general d

- $\mathbf{z}_*(\overline{\mathbf{r}})$ turns out to be a critical point for $\overline{\mathbf{r}}$ iff the outward normal to $\log \mathcal{V}$ is parallel to $\mathbf{r}$. In other words, for some $\lambda \in \mathbb{C}$, $\mathbf{z}_*$ solves

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- Then

$$a_{\mathbf{r}} \sim \mathbf{z}_*(\overline{\mathbf{r}})^{-\mathbf{r}} \sqrt{\frac{1}{(2\pi|\mathbf{r}|)^{(d-1)/2} \kappa(\mathbf{z}_*)}} \frac{G(\mathbf{z}_*)}{|\nabla_{\log} H(\mathbf{z}_*)|}$$

where $|\mathbf{r}| = \sum_i r_i$ and κ is the Gaussian curvature of $\log \mathcal{V}$ at $\log \mathbf{z}_*$.

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- The Gaussian curvature can be computed explicitly in terms of derivatives of H to second order.

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- Compare Panholzer-Prodinger, Bull. Aust. Math. Soc. 2012.

Multiple points: generic shape of $\mathcal{A}(\mathbf{z}_*)$

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$$G(\mathbf{z}_*)P\left(\frac{r_1}{z_1^*}, \dots, \frac{r_d}{z_d^*}\right),$$

  P a piecewise polynomial of degree $n - d$.

Example (Queueing network)

- Consider

$$F(x,y) = \frac{\exp(x+y)}{(1-\frac{2x}{3}-\frac{y}{3})(1-\frac{2y}{3}-\frac{x}{3})}$$

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- Note we say nothing here about the boundary of the cone.

Enumerating lattice walks confined to the first quadrant

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- Stephen Melczer & MCW (2016): confirmation of asymptotics of 19 remaining cases (correcting constants in some cases).

Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	equiv		OEIS	$\mathfrak{S}$	alg	equiv
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

Easy generalizations

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- A toral point is one for which every point on its torus is a minimal singularity, such as $1/(1 - x^2y^3)$. We deal with this by an easy modification of the reduction to the F-L integral.
- If the dominant point is smooth but H is not locally squarefree, then we obtain polynomial corrections that are easily computed (take higher derivative when computing the residue).

Example (Periodicity)

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.
- There is also a dominant point at $-\mathbf{p}$. Adding the contributions yields the correct asymptotic:
$$a_{rs} \sim \sqrt{\frac{2}{\pi}}(-1)^{(s-r)/2}\left(\frac{2r}{\sqrt{s^2-r^2}}\right)^{-r}\left(\sqrt{\frac{s-r}{s+r}}\right)^{-s}\sqrt{\frac{s+r}{r(s-r)}}$$
when $r+s$ is even, and zero otherwise.

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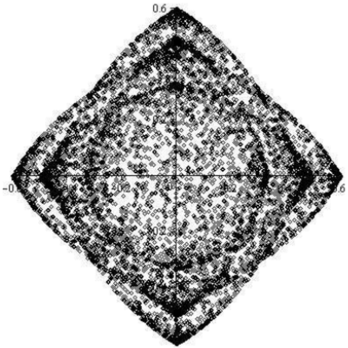
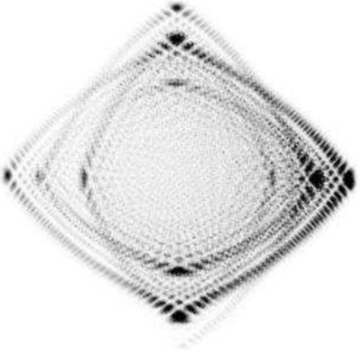
- On $\mathcal{V}$, $|x_i| = 1$ for all i implies $|y| = 1$, so we get more poles than expected.
- The set of *feasible velocities* is the region of non-exponential decay of amplitudes, which we can approximate very well – see next slide.

Feasible region for 2-D QRW (L: theory, R: time 200)





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- We have dealt with a class of cone singularities arising in statistical physics problems, but the analysis is much harder.
- If the geometry changes, we typically encounter a phase transition.

Example (nonpositive case - PhD thesis Tim DeVries)

- Consider *bicolored supertrees*

$$F(x,y) = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}.$$

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- In fact $(2, 1/8)$ dominates.
- The answer:

$$a_{nn} \sim \frac{4^n\sqrt{2}\Gamma(5/4)}{4\pi} n^{-5/4}.$$

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- In directions away from $n = 3k$, our ordinary smooth point analysis holds. When $n = 3k$ we can redo the F-L integral easily and obtain asymptotics of order $n^{-1/3}$.
- Determining the behaviour as we approach this diagonal at a moderate rate is harder (Manuel Lladser PhD thesis), and recovers the results of Banderier-Flajolet-Schaeffer-Soria 2001.

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- We can in principle differentiate implicitly and solve a system of equations for each term in the asymptotic expansion.
- Hörmander has a completely explicit formula that proved useful. There may be other ways.

Hörmander's explicit formula

For an isolated nondegenerate stationary point in dimension d,

$$I(\lambda) \sim \left(\det\left(\frac{\lambda f''(\mathbf{0})}{2\pi}\right)\right)^{-1/2} \sum_{k\geq 0} \lambda^{-k} L_k(A, f)$$

where L_k is a differential operator of order $2k$ evaluated at $\mathbf{0}$. Specifically,

$$\underline{f}(t) = f(t) - (1/2)tf''(0)t^T$$

$$\mathcal{D} = \sum_{a,b} (f''(\mathbf{0})^{-1})_{a,b}(-\mathrm{i}\partial_a)(-\mathrm{i}\partial_b)$$

$$L_k(A, f) = \sum_{l\leq 2k} \frac{\mathcal{D}^{l+k}(A\underline{f}^l)(0)}{(-1)^k 2^{l+k} l!(l+k)!}.$$

Example (2 planes in 3-space)

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- The line of intersection of the two planes supplies the other directions. Each point on the line $\{(1, 1, 1) + \lambda(-1, -1, -3) \mid -1/3 < \lambda < 1\}$ gives asymptotics in a 2-D cone.

Example (2 planes in 3-space, continued)

Using the formula we obtain

$$a_{3t,3t,2t} = \frac{1}{\sqrt{3\pi}}\left(\frac{1}{4}t^{-1/2} - \frac{25}{1152}t^{-3/2} + \frac{1633}{663552}t^{-5/2}\right) + O(t^{-7/2}).$$

rel. err.	1	2	4	8	16	32
$k=1$	-0.660	-0.315	-0.114	-0.0270	-0.00612	-0.00271
$k=2$	-0.516	-0.258	-0.0899	-0.0158	-0.000664	0.00000780
$k=3$	-0.532	-0.261	-0.0906	-0.0160	-0.000703	-0.00000184

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- Example: suppose $x^3 - x^2 + 11x - 2 = 0, x > 0$, and we want $g(x) := x^5/(867x^4 - 1)$.

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- Minimal polynomial is $11454803y^3 - 2227774y^2 + 2251y - 32$.

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- Unfortunately, computations in the local ring are not effective (as far as we know). If a polynomial factors as an analytic function, but the factors are not polynomial, we can't deal with it algorithmically (yet).
- Smooth points are easily detected. There are some sufficient conditions, and some necessary conditions, for $\mathbf{z}_*$ to be a multiple point. But in general we don't know how to classify singularities algorithmically.

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- Reduction to the squarefree case is then easy and algorithmic.

Summary

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- The basic smooth and multiple point results have been incorporated into Sage. There is work to do on better symbolic algorithms and implementation in software.
- Algebraic GFs are still largely unexplored. We know how to reduce to the rational case, but at the cost of needing higher order terms and leaving the positive case.
- Convex analysis suffices for most combinatorial applications so far, but more geometry and topology will be needed to make serious progress beyond the positive case.

Main references

- R. Pemantle and M.C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge University Press 2013. https://www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/

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- R. Pemantle and M.C. Wilson, *Twenty Combinatorial Examples of Asymptotics Derived from Multivariate Generating Functions*, SIAM Review 2008.
- Sage implementation by Alex Raichev: package `asymptotics_multivariate_generating_functions`.

Safonov's basic construction

- Suppose that F is algebraic and its defining polynomial P satisfies
$$P(w, \mathbf{z}) = (w - F(\mathbf{z}))^k u(w, \mathbf{z})$$
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- Define
$$R(z_0, \mathbf{z}) = \frac{z_0^2 P_1(z_0, z_0 z_1, z_2, \dots)}{k P(z_0, z_0 z_1, z_2, \dots)}$$
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- Higher order terms are essential: the numerator of $\tilde{R}$ always vanishes at the dominant point.

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- Definition: Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have $d+1$ variables and let M be a $d \times d$ matrix with nonnegative entries. The M-diagonal of F is the formal power series in d variables whose coefficients are given by $b_{r_2,\dots r_d} = a_{s_1,s_1,s_2,\dots s_d}$ and $(s_1, \dots, s_d) = (r_1, \dots, r_d)M$.

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- Theorem: Let f be an algebraic function of d variables. Then there is a unimodular integer matrix M with positive entries and a rational function F in $d+1$ variables such that f is the M-diagonal of F.
- The example $x\sqrt{1-x-y}$ shows that the elementary diagonal cannot always be used.

Example (Narayana numbers)

- The bivariate GF $F(x,y)$ for the Narayana numbers

$$a_{rs} = \frac{1}{r}\binom{r}{s}\binom{r-1}{s-1}$$

satisfies $P(F(x,y),x,y) = 0$, where

$$\begin{aligned} P(w,x,y) &= w^2 - w\,[1 + x(y-1)] + xy \\ &= [w - F(x,y)]\,\left[w - \overline{F}(x,y)\right]. \end{aligned}$$

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- Using the above construction we obtain the lifting

$$G(u, x, y) = \frac{u(1 - 2u - ux(1-y))}{1 - u - xy - ux(1-y)}.$$

Example (Narayana numbers continued)

- The above lifting yields asymptotics by smooth point analysis in the usual way. The critical point equations yield

$$u = s/r, x = \frac{(r-s)^2}{rs}, y = \frac{s^2}{(r-s)^2}.$$

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- Interestingly, specializing $y = 1$ commutes with lifting. Is this always true?

Example (Partition function for BPS operators)

- Consider

$$F(x, y) = \frac{1}{\prod_{i=1}^{\infty} (1 - x^i - y^i)}$$

for which we seek diagonal asymptotics. The singular variety is in fact smooth at the relevant points.

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- Although poles accumulate on the unit torus, they are not in the right place to cause trouble.
- In progress with A. Zahabi (Queen Mary, London).

Example (Green's function for face-centred cubic lattice)

- Consider

$$F(x,y) = \frac{1}{(1 - z x_1 \cdots x_d)\left(1 - \frac{z}{|S|} x_1 \cdots x_d \lambda(\mathbf{x})\right)}$$

where

$$\lambda(\mathbf{x}) = \sum_{\mathbf{r} \in S} \mathbf{x}^{\mathbf{r}}$$

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Reduction step 1: localization

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$$a_{rs} = (2\pi i)^{-2} \int_{C_\rho} z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z, w) \frac{dw}{w} \frac{dz}{z}.$$

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- The inner integral is small away from z_*, so that for some small neighbourhood N of z_* in C_ρ,

$$a_{rs} \approx I := (2\pi i)^{-2} \int_N z^{-r} \int_{C_{\sigma-\delta}} w^{-s} F(z,w) \frac{dw}{w} \frac{dz}{z}.$$

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- Note that this is because of strict minimality: off N, the function $F(z,\cdot)$ has radius of convergence greater than σ, and compactness allows us to do everything uniformly.

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$$I = I' + (2\pi i)^{-1} v(z)^s \Psi(z),$$

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- Clearly $|z_*^r I'| \to 0$, and hence
$$a_{rs} \approx (2\pi i)^{-1} \int_N z^{-r} v(z)^s \Psi(z)\, dz.$$

Reduction step 3: Fourier-Laplace integral

- We make the substitution

$$
\begin{aligned}
f(\theta) &= -\log \frac{v(z_* e^{i\theta})}{v(z_*)} + i\frac{r\theta}{s} \\
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Reduction step 3: Fourier-Laplace integral

- We make the substitution

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- This yields

$$a_{rs} \sim \frac{1}{2\pi} z_*^{-r} w_*^{-s} \int_D \exp(-sf(\theta)) A(\theta)\, d\theta$$

where D is a small neighbourhood of $0 \in \mathbb{R}$.

Example (local factorization of lemniscate)

- Given $F = 1/H$ where $H(x, y) =$
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- Note that our double point formula does not require details of the individual factors. However this is not the case for general multiple points.