Embedding algebraic generating functions into rational ones

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AIM ACSV 2022

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- Steve, Robin and I are writing the second edition of *Analytic Combinatorics in Several Variables*.
- We aim to substantially increase uptake of ACSV by end users.
- Your feedback will be useful. Please give it!
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Algebraic GFs

- All GFs in this presentation are formal power series (some things extend to Laurent series but we do not consider that here).
- The ACSV project has extensive results for asymptotic coefficient extraction from rational generating functions in fixed dimension.
- Algebraic functions that are not rational occur in many natural problems. In the univariate case, there is extensive effective asymptotic theory (e.g. Flajolet-Sedgewick book; Michael Drmota (here)).
- In general dimension,Torin Greenwood derived asymptotics for special algebraic singularities, with very considerable work, using custom contours.
- Can we instead treat algebraic functions by reducing to the rational case, and using rational ACSV?

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Diagonals

- We need to define diagonal more generally. Let

$$F(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}}$$

be a d-variate generating function and $\mathbf{x}^o = (x_2, x_3, \ldots, x_d)$.

- The elementary diagonal is

$$\Delta F(\mathbf{x}^o) = \sum_{\mathbf{r}} a_{r_2, r_2, r_3, \ldots, r_d} x_2^{r_2} x_3^{r_3} \ldots x_d^{r_d}.$$

- Any composition of this operation with a permutation of variables is called a diagonal of F, and the *leading diagonal* is the GF $\operatorname{diag} F$ obtained by reducing to 1 variable.
- Old conjecture: every univariate D-finite GF (with mild necessary conditions) is the leading diagonal of some rational function.

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Example (Multinomial coefficients)

Suppose

$$F(x,y,z) = \sum_{r,s,t} \frac{(r+s+t)!}{r!s!t!} x^r y^s z^t.$$

Then

$$\Delta F(y,z) = \sum_{s,t} \frac{(2s+t)!}{s!^2 t!} y^s z^t$$

and

$$\Delta\Delta F(z) = \operatorname{diag} F(z) = \sum_{t} \frac{(3t)!}{t!^3} z^t.$$

Theorem (Furstenberg; Hautus & Klarner)

If $F(x, y)$ is a rational power series in $d = 2$ variables then $\operatorname{diag} F$ is algebraic.

Proof sketch.

Since $F = P/Q$ converges in a neighborhood of the origin, when $|x|$ is sufficiently small the function $F(x/y, y)$ is absolutely convergent for y in some annulus $A(x)$.
The constant term in y of $F(x/y, y)$ equals $\operatorname{diag} F(x)$.
Cauchy's integral formula yields, where C is any positively oriented circle in $A(x)$,

$$\begin{aligned}\operatorname{diag} F(x) &= \frac{1}{2\pi i} \int_C \frac{P(x/y, y)}{yQ(x/y, y)}\, dy \\ &= \sum_{y=\alpha(x)} \operatorname{Res}[F(x/y, y); y].\end{aligned}$$

Going the other way (*embedding*)

Theorem (Furstenberg, Safonov)

Let $f(x)$ *be an algebraic* d*-variate power series with* $P(\mathbf{x}, f(\mathbf{x})) = 0$ *for some* $P(\mathbf{x}, y) \in \mathbb{C}[\mathbf{x}, y]$*. Suppose further that* $f(0, x_2, \ldots, x_d) = 0$ *and* $P_y(\mathbf{0}, 0) \neq 0$*. Then*

$$f(x) = \Delta \frac{y^2 P_y(x_1 y, x_2, \ldots, x_d, y)}{P(x_1 y, x_2, \ldots, x_d, y)}.$$

Proof sketch.

The hypotheses imply that there is a single branch of the algebraic function through the origin, and it can be computed as a residue using the Argument Principle, inverting the diagonal extraction above. □

Example

The Narayana GF $\sum_{r,s} a_{rs} x^r y^s$ embeds in

$$F(u,x,y) = \frac{u(1-2u-ux(1-y))}{1-u-xy-ux(1-y)} = \sum_{r,s,t} b_{trs} u^t x^r y^s.$$

and $a_{rs} = b_{rrs}$.
Also putting $y = 1$ yields shifted Catalan GF
$\sum_n c_n x^n = (1-\sqrt{1-4x})/2$ which embeds in

$$F(u,x) = \frac{u(1-2u)}{(1-u-x)}$$

so specialization commutes with embedding for this method.
To derive asymptotics when $r = \alpha n, s = \beta n$, we consider the direction determined by (α, α, β). Smooth point analysis works nicely!

General results on embedding

- Denef & Lipshitz (1987): embedding into dimension $2d$
- Adamczewski & Bell (2013): effective embedding into dimension $2d$
- Denef & Lipshitz (1987): unimodular change of indices, embedding into dimension $d+1$
- Safonov (2000); unimodular change of indices, effective embedding into dimension $d+1$

They all require separation of the branches of the algebraic function, hence resolution of singularities. All make various changes of variable to reduce to the single branch ("étale") case.

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ACSV issues

- The rational functions R obtained by the above methods are not in general combinatorial.
- They often have a contributing critical point at infinity.
- The leading term in our asymptotic formulae vanishes.
- If we could ensure that R is combinatorial, it would help considerably.

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Example (Safonov)

Let

$$f(x,y) = x\sqrt{1-x-y}.$$

Then F is not the elementary diagonal of any 3-variable rational GF. However, we have

$$\begin{aligned} F(u,x,y) &= \frac{ux(2+x+xy+2u^2+3u)}{2+u+x+xy} \\ &= \sum_{t,r,s} a_{trs} u^t x^r y^s \end{aligned}$$

and

$$f(x,y) = \sum_{r,s} a_{r+s,r+s,s} x^r y^s.$$

Example (simplest univariate with more than one branch)

Let $f(x) = x/\sqrt{1-x}$. This is *combinatorial*, having all nonnegative coefficients. Also $f = \operatorname{diag} F$ where

$$F(x,u) = \frac{2xu}{2+x+u}$$

Note that $(x,y) \mapsto (-x,-y)$ shows that $f = \operatorname{diag} \overline{F}$ where

$$\overline{F}(x,u) = \frac{2xu}{2-x-u}$$

and F is combinatorial.

However, the above-mentioned methods yield different embeddings. For example Safonov gives

$$\left(\frac{2\,(xu-1)(u+1)u}{xu^2 + 2\,xu + x - u - 2} + 1 \right) xu.$$

Research questions

- What is the deeper connection between the different F each yielding $\operatorname{diag} F = f$?
- Can we always choose R to be combinatorial and “nice” (amenable to ACSV)?
- When do we get critical points at infinity? Can we deal with them easily?

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